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Bipartition of graphs based on the normalized cut and spectral methods

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Abstract. In the first part of this paper, we survey results that are associated with three types of Laplacian matrices:difference, normalized, and signless. We derive eigenvalue and eigenvector formulaes for paths and cycles using circulant matrices and present an alternative proof for finding eigenvalues of the adjacency matrix of paths and cycles using Chebyshev polynomials. Even though each results is separately well known, we unite them, and provide uniform proofs in a simple manner. The main objective of this study is to solve the problem of finding graphs, on which spectral clustering methods and normalized cuts produce different partitions. First, we derive a formula for a minimum normalized cut for graph classes such as paths, cycles, complete graphs, double-trees, cycle cross paths, and some complex graphs like lollipop graph $LP_{n,m}$, roach type graph $R_{n,k}$, and weighted path $P_{n,k}$. Next, we provide characteristic polynomials of the normalized Laplacian matrices $\mathcal{L}(P_{n,k})$ and $\mathcal{L}(R_{n,k})$. Then, we present counter example graphs based on $R_{n,k}$, on which spectral methods and normalized cuts produce different clusters.

Keywords. spectral clustering, normalized Laplacian matrices, difference Laplacian matrices, signless Laplacian matrices, normalized cut

1. Introduction

Clustering techniques are common in multivariate data analysis, data mining, machine learning, and so on. The goal of the clustering or partitioning problem is to find groups such that entities within the same group are similar and different groups are dissimilar. In the graph-partitioning problem, much attention is given to find the precise criteria to obtain a good partition. Clustering methods that use eigenvalues and eigenvectors of matrices associated with graphs are called spectral clustering methods and are widely used in graph- partitioning problems. In particular, eigenvalues and eigenvectors of Laplacian matrices play a vital role in graph-partitioning problems. In 1973, Fiedler defined the second smallest eigenvalue λ_2 of a difference Laplacian matrix as the algebraic connectivity of a graph [7]. In 1975, he showed that we can decompose a graph G into two connected components by only using the sign structure of an eigenvector related to the second smallest eigenvalue [8]. In 2001, Fiedler's investigation was extended by Davies using the discrete nodal domain theorem [5]. Laplacian, normalized Laplacian, and adjacency matrices with negative entries can be used with the nodal domain theorem. This theorem is useful to identify the number of connected sign graphs of a given graph on the basis of their eigenvectors and eigenvalues.

In 1984, Buser [3] investigated the graph invariant quantity $i(G) = \min_U \frac{|\partial U|}{|U|}$, which considers the relationship between size of a cut and the size of a separated subset U. He defined the isoperimetric number $i(G)$, and the optimal bisection was given by the minimum $i(G)$. Guattery and Miller [9, 10] considered two spectral separation algorithms that partition the vertices on the basis of the values of their corresponding entries in the second eigenvector and, in 1995, they provided some counter examples for which each of these algorithms produce poor separators. They used an eigenvector based on the second smallest eigenvalue of a difference Laplacian matrix as well as a specified number of eigenvectors corresponding to the smallest eigenvalues. Finally, they extended it to the generalized version of spectral methods that allows for the use of more than a constant number of eigenvectors and showed that there are some graphs for which the performance of all the above spectral algorithms was poor. We follow their methods especially in the cases of graph automorphism and even -odd eigenvector theorem for the concrete classes of graphs such as roach graphs, double-trees, and double-tree cross paths. We prefer to use a normalized Laplacian matrix rather than a difference Laplacian matrix, and describe these properties in terms of formal graph notation.

In 1997, Fan Chung [4] discussed the most important theories and properties regarding eigenvalues of normalized Laplacian matrices and their applications to graph separator problems. She considered the partitioning problem using Cheeger constants and derived fundamental relations between the eigenvalues and Cheeger constants. In 2000, Shi and Malik [14] proposed a measure of disassociation, called normalized cut, for the image segmentations. This measure computed the cut cost as a fraction of total edge connections. The normalized cut is used to minimize the disassociation between groups and maximize the association within groups. However, minimization of normalized cut criteria is an non-deterministic polynomial-time hard (NP- hard) problem. Therefore, approximate discrete so-

lutions are required. The solution to the minimization problem of the normalized cut is given by the second smallest eigenvector of the generalized eigensystem, $(D-W)y = \lambda Dy$, where D is the diagonal matrix with vertex degrees and W is a weighted adjacency matrix. Shi and Malik used a minimum normalized cut value as a splitting point and found a bisection using the second smallest eigenvector. They realized that the eigenvectors are well separated and that this type of splitting point is very reliable. The normalized cut introduced by Shi and Malik [14] is useful in several areas. This measure is of interest not only for image segmentation but also for network theories and statistics [1, 13, 12, 6, 15].

In this study, we review the known results regarding the difference, normalized, and signless Laplacian matrices. Then, we give uniform proofs for the eigenvalues and eigenvectors of paths and cycles. Next, we analyze the minimum normalized cut from the view point of connectivity of graphs and compare the results with those of the spectral bisection method. Special emphasis is given to classify the graphs, that poorly perform on spectral bisections using normalized Laplacian matrices. We use the term $Mcut(G)$ to represent the minimum normalized cut and $Lcut(G)$ to represent the normalized cut of the bipartition created by the second smallest eigenvector of the normalized Laplacian based on the sign pattern. Finding $Mcut(G)$ for a graph is NP-hard. However, we derive a formula for $Mcut(G)$ for some basic classes of graphs such as paths, cycles, complete graphs, double-trees, cycle cross paths, and some complex graphs like lollipop type graphs $LP_{n,m}$, roach type graphs $R_{n,k}$ and weighted paths $P_{n,k}$. Next, we present characteristic polynomials of the normalized Laplacian matrices $\mathcal{L}(P_{n,k})$ and $\mathcal{L}(R_{n,k})$. We provide counter example graphs on the basis of a graph $R_{n,k}$ on which $Mcut(G)$ and $Lcut(G)$ have different values.

This paper is organized as follows. In section 2, we present basic terminologies and key results related to the difference, normalized, and signless Laplacian matrices. In particular, we summarize the upper and lower bounds of the second smallest eigenvalues. We also define graphs that are used in other sections using formal notation. In section 3, we review the properties of the $Mcut(G)$ of graphs and derive formulae for the $Mcut(G)$ of some basic classes of graphs and some complex graphs such as $R_{n,k}$, $P_{n,k}$, and $LP_{n,m}$. In section 4, we consider the eigenvalues and eigenvectors of paths and cycles for the three types of Laplacian matrices introduced above. In particular, we review the eigenvalue formulae for the three types of Laplacian matrices using circulant matrices and then review an alternative proof for the eigenvalues of adjacency matrices of paths and cycles using Chebyshev polynomials. We also give concrete formulae for the characteristic polynomials of the normalized Laplacian matrices $\mathcal{L}(P_{n,k})$ and $\mathcal{L}(R_{n,k})$. In section 5, we provide counter example graphs for which spectral techniques perform poorly compared with the normalized cut. Specifically, we find the conditions for which $Mcut(G)$ and $Lcut(G)$ have different values on the $R_{n,k}$ graph.

2. Preliminaries

An undirected graph is an ordered pair $G = (V(G), E(G))$, where $V(G)$ is a finite set, elements of which are called vertices, and we represent $V(G)$ as $V(G) = \{v_1, v_2, \ldots, v_n\}$. $E(G)$ is a set of two-element subsets of $V(G)$, called edges. Conventionally, we denote an edge $\{v_i, v_j\}$ by (v_i, v_j) in this paper. Two vertices v_i and v_j of G are called adjacent, if $(v_i, v_j) \in E(G)$. For simplicity, sometimes we use V instead of $V(G)$ and E instead of $E(G)$. The number of vertices in G is the order of G and the number of edges is the size of G. For a given subset $S \subseteq V$, $|S|$ represent the size of the set S. For a subset $A \subseteq V$, we represent the set of vertices not belongs to A as $V \setminus A = \{v_i \mid v_i \notin A\}$. A graph of order 1 is called a trivial graph. A graph which has two or more vertices is called a nontrivial graph. A graph of size 0 is called an empty graph. Assume that all graphs in this paper are finite, undirected and have edge weight 1.

Definition 1 (Adjacency matrix)**.** Let $G = (V, E)$ be a graph and $|V| = n$. The adjacency matrix $A(G) = (a_{ij})$ of an undirected graph G is a $n \times n$ matrix whose entries are given by

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2 (Degree)**.** The degree d_i of a vertex v_i of a graph G is defined as $d_i = \displaystyle\sum_{j=1}^{n} a_{ij}$. Minimum and maximum degree of a graph G are denoted by $\delta(G)$ and $\triangle(G)$, respectively.

Definition 3 (Degree Matrix)**.** The diagonal matrix of a graph G is denoted by $D(G) = diag(d_1, d_2, \ldots, d_n)$, where d_i is the degree of a vertex v_i.

Note: For simplicity, sometimes we use D instead of $D(G)$.

Definition 4 (Volume)**.** The volume of a graph $G = (V, E)$ denoted by $vol(G) = \displaystyle\sum_{i=1}^{|V|} d_i$, is the sum of the degrees of vertices in V. The volume of a subset $A \subset V$ is denoted by $vol(A) = \displaystyle\sum_{i \in A} d_i$.

Definition 5 (Edge Connectivity)**.** The edge connectivity of a graph G is denoted by $\kappa'(G)$, is the minimum number of edges needed to remove in order to disconnect the graph. A graph is called k-edge connected if every disconnecting set has at least k edges. A 1-edge connected graph is called a connected graph.

Definition 6 (Cartesian product)**.** The Cartesian product of graphs G and H is denoted by $G \square H = (V(G \square H), E(G \square H))$, where $V(G \square H) = V(G) \times V(H)$ and $E(G \square H) = \{(u_1, v_1), (u_2, v_2) \mid u_1 = u_2 \text{ and } (v_1, v_2) \in E(H) \text{ or } v_1 = v_2 \text{ and } (u_1, u_2) \in E(G)\}$.

We note that $G_1 \square G_2 \cong G_2 \square G_1$, $\delta(G_1 \square G_2) = \delta(G_1) + \delta(G_2)$, and $\kappa'(G \square H) = \min\{\kappa'(G)|V(H)|, \kappa'(H)|V(G)|, \delta(G) + \delta(H)\}$.

Definition 7 (Path)**.** Let $G = (V, E)$ be a graph. A path in a graph is a sequence of vertices such that from each of its vertices there is an edge to the next vertex in the sequence. This is denoted by $P = (u = v_0, v_1, \ldots, v_k = v)$, where $(v_i, v_{i+1}) \in E$ for $0 \leq i \leq k-1$. The length of the path is the number of edges encountered in P.

Definition 8 (Shortest Path)**.** Let $G = (V, E, w)$ be a weighted graph. Let P be a set of paths from vertex i to j. Denote $\ell(p)$, the length of the path $p \in P$. Then p is a shortest path if $\ell(p) = \min_{p' \in P} \ell(p')$.

Definition 9 (Distance)**.** The distance between two vertices $i, j \in V$ of the graph G is denoted by $dist(i, j)$ is the length of a

shortest path between vertex i and j.

Definition 10 (Diameter)**.** The diameter of a graph $G = (V, E)$ is given by $diam(G) = \max\{dist(i, j) \mid i, j \in V\}$.

Definition 11 (Permutation matrix)**.** Let $G = (V, E)$ be a graph. The permutation ϕ defined on V can be represented by a permutation matrix $P = (p_{ij})$, where

$$p_{ij} = \begin{cases} 1 & \text{if } v_i = \phi(v_j), \\ 0 & \text{otherwise.} \end{cases}$$

Definition 12 (Automorphism)**.** Let $G = (V, E)$ be a graph. Then a bijection $\phi : V \to V$ is an automorphism of G if $(v_i, v_j) \in E$ then $(\phi(v_i), \phi(v_j)) \in E$. In other words automorphisms of G are the permutations of vertex set V that maps edges onto edges.

Proposition 1 (Biggs [2])**.** *Let $A(G)$ be the adjacency matrix of a graph $G = (V, E)$, and P be the permutation matrix of permutation ϕ defined on V. Then ϕ is an automorphism of G if and only if $PA = AP$.*

□

Definition 13 (Weighted graph)**.** A weighted graph is denoted by $G = (V, E, w)$, where $w : E \to \Re$.

Definition 14 (Weighted adjacency matrix)**.** The weighted adjacency matrix $W = (w_{ij})$ is defined as

$$w_{ij} = \begin{cases} w(i, j) & \text{if } (i, j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

The degree d_i of a vertex v_i of a weighted graph is defined by $d_i = \sum_{j=1}^{n} w_{ij}$. Unweighted graphs are special cases, where all edge weights are 0 or 1.

Definition 15 (Graph cut)**.** A subset of edges which disconnects the graph is called a graph cut. Let $G = (V, E, w)$ be a weighted graph and $W = (w_{ij})$ the weighted adjacency matrix. Then for $A, B \subset V$ and $A \cap B = \emptyset$, the graph cut is denoted by $cut(A, B) = \sum_{i \in A, j \in B} w_{ij}$.

Definition 16 (Isoperimetric number)**.** The isoperimetric number $i(G)$ of a graph G of order $n \geq 2$ is defined as

$$i(G) = \min\{\frac{cut(S, V \setminus S)}{|S|}, S \subset V, 0 < |S| \leq \frac{n}{2}\}.$$

Definition 17 (Cheeger Constant-edge expansion)**.** Let $G = (V, E)$ be a graph. For a nonempty subset $S \subset V$, define $h_G(S) = \dfrac{cut(S, V \setminus S)}{\min(vol(S), vol(V \setminus S))}$. The Cheeger constant(edge expansion) h_G is defined as $h_G = \min_S h_G(S)$.

Definition 18 (Cheeger constant-vertex expansion)**.** Let $G = (V, E)$ be a graph. For a nonempty subset $S \subset V$, define $g_G(S) = \dfrac{vol(\delta S)}{\min(vol(S), vol(V \setminus S))}$, where $\delta S = \{v \notin S : (u, v) \in E, u \in S\}$. Then the Cheeger constant(vertex expansion) g_G is defined as $g_G = \min_S g_G(S)$.

Definition 19 (Weighted difference Laplacian)**.** The **weighted difference Laplacian** $L(G) = (l_{ij})$ is defined as

$$l_{ij} = \begin{cases} d_i - w_{ii} & \text{if } v_i = v_j, \\ -w_{ij} & \text{if } v_i \text{ and } v_j \text{ are adjacent and } v_i \neq v_j, \\ 0 & \text{otherwise.} \end{cases}$$

This can be written as $L(G) = D(G) - W(G)$.

Definition 20 (Weighted normalized Laplacian)**.** The **weighted normalized Laplacian** $\mathcal{L}(G) = (\ell_{ij})$ is defined as

$$\ell_{ij} = \begin{cases} 1 - \frac{w_{jj}}{d_j} & \text{if } v_i = v_j, \\ -\frac{w_{ij}}{\sqrt{d_i d_j}} & \text{if } v_i \text{ and } v_j \text{ are adjacent and } v_i \neq v_j, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1. *Let G be a graph, n the size of graph G, $A = (w_{ij})$ a weighted adjacency matrix of G, λ an eigenvalue of $\mathcal{L}(G)$ and x an eigenvector corresponding to λ with $x^T x = 1$. Then,*

$$\lambda = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{x_i}{\sqrt{d_i}} - \frac{x_j}{\sqrt{d_j}} \right)^2 w_{ij}.$$

Proof. Let D be the degree matrix of G. The normalized Laplacian matrix $\mathcal{L}(G)$ is defined by $D^{-\frac{1}{2}}(D - A)D^{-\frac{1}{2}}$. Let y be a vector with size n and $x = D^{\frac{1}{2}} y$. Then $x^T \mathcal{L}(G) x = \left(D^{\frac{1}{2}} y\right)^T \mathcal{L}(G) \left(D^{\frac{1}{2}} y\right) = y^T D^{\frac{1}{2}} \mathcal{L}(G) D^{\frac{1}{2}} y = y^T (D - A) y = \sum_{i=1}^{n} y_i^2 d_i - \sum_{i=1}^{n} \sum_{j=1}^{n} y_i y_j w_{ij} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(y_i - y_j\right)^2 w_{ij} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{x_i}{\sqrt{d_i}} - \frac{x_j}{\sqrt{d_j}} \right)^2 w_{ij}$. Since x is an an eigenvector of $\mathcal{L}(G)$ corresponding to λ and $x^T x = 1$, we have $\lambda = \dfrac{x^T(\lambda x)}{x^T x} = \dfrac{x^T(\mathcal{L}(G) x)}{x^T x} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{x_i}{\sqrt{d_i}} - \frac{x_j}{\sqrt{d_j}} \right)^2 w_{ij}$. □

There are several properties about bounds of the second eigenvalue λ_2.

Proposition 2 (Mohar[11])**.** *Let $G = (V, E)$ be a graph and λ_2 be the second smallest eigenvalue of $L(G)$. Then,*

$$\frac{\lambda_2}{2} \leq i(G) \leq \sqrt{(2\triangle(G) - \lambda_2)\lambda_2}.$$

□

Proposition 3 (Chung[4])**.** *Let G be a connected graph and h_G the Cheeger constant of G. Then,*

1. $\dfrac{2}{vol(G)} < h_G$,
2. $1 - \sqrt{1 - h_G^2} < \lambda_2$, *and*
3. $\dfrac{h_G^2}{2} < \lambda_2 \leq 2h_G$.

□

Matrix M	$M(P_4)$
Adjacency $A(P_4)$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$
Difference Laplacian $L(P_4)$	$\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$
Normalized Laplacian $\mathcal{L}(P_4)$	$\begin{pmatrix} 1 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 1 \end{pmatrix}$
Signless Laplacian $SL(P_4)$	$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

Table 1: Matrices associated with graphs.

Definition 21 (Signless Laplacian)**.** The **weighted signless Laplacian** $SL(G) = (sl_{ij})$ is defined as

$$sl_{ij} = \begin{cases} d_i + w_{ii} & \text{if } v_i = v_j, \\ w_{ij} & \text{if } v_i \text{ and } v_j \text{ are adjacent and } v_i \neq v_j, \\ 0 & \text{otherwise.} \end{cases}$$

This can be written as $SL(G) = D + W$.

Definition 22 (Path graph)**.** A path graph $P_n = (V_n, E_n)$ consists of a vertex set $V_n = \{v_l \mid l \in \mathbb{Z}^+, l \leq n\}$ and an edge set $E_n = \{(v_l, v_{l+1}) \mid 1 \leq l < n\}$.

Example 1. The Table 1 shows an adjacency matrix and the three Laplacian matrices discussed in the above for path graph P_4.

Lemma 2. *Let $G = (V, E, w)$ be a weighted graph. Then the eigenvalues of $\mathcal{L}(G)$ and $D^{-1}L(G)$ are equal.*

Proof. $D^{-1}L = D^{-1}(D - W) = I - D^{-1}W = D^{-1/2}DD^{-1/2} - D^{-1/2}D^{-1/2}W = D^{-1/2}(D - W)D^{-1/2}$. Therefore $D^{-1}L(G) = \mathcal{L}(G)$ and has the same spectrum. □

Definition 23 (Regular graph)**.** A graph $G = (V, E)$ is called r-regular, if $d_i = r$ $(i = 1, \ldots, |V|)$.

Lemma 3. *Let $\mu_i, (i = 1, \ldots, n)$ be eigenvalues of difference Laplacian matrix $L(G) = D(G) - A(G)$. Then for any regular graph of degree r, normalized Laplacian eigenvalues are $\lambda_i = \frac{\mu_i}{r}, (i = 1, \ldots, n)$.*

Proof. $L = (D - A) = rI - A$. Then $\mathcal{L}(G) = D^{-1/2}LD^{-1/2} = \frac{I}{r^{1/2}}(rI - A)\frac{I}{r^{1/2}} = I - \frac{A}{r}$. Then $r\mathcal{L}(G) = L(G)$. If μ_i is an eigenvalue of L then it is an eigenvalue of $r\mathcal{L}(G)$. This shows that $\lambda(\mathcal{L}(G)) = \frac{\mu_i}{r}(i = 1, \ldots, n)$. □

Proposition 4. *Let $\mathcal{L}(G)$ be the normalized Laplacian matrix of a graph G and P be the permutation matrix corresponding to the automorphism ϕ defined on V. If U is an eigenvector of $\mathcal{L}(G)$ with an eigenvalue λ, then PU is also an eigenvector of $\mathcal{L}(G)$ with the same eigenvalue.*

Proof. From the definition of automorphism $P^T\mathcal{L}(G)P = \mathcal{L}(G)$. Then $\mathcal{L}(G)U = \lambda U$ implies that $(P^T\mathcal{L}(G)P)U = \lambda U$. Since $PP^T = I$, we get $\mathcal{L}(G)PU = \lambda(PU)$. If U is an eigenvector of $\mathcal{L}(G)$ with an eigenvalue λ then PU is also an eigenvector with the same eigenvalue. □

Remarks. This result holds for any matrix associated with a graph under the automorphism defined on a vertex set.

Definition 24 (Odd-even vectors)**.** Let $G = (V, E)$ be a graph and $\phi : V \to V$ be an automorphism of order 2. A vector x is called an even vector if $x_i = x_{\phi(i)}$ for all $1 \leq i \leq n$ and a vector y is called an odd vector if $y_i = -y_{\phi(i)}$ for all $1 \leq i \leq n$, where $n = |V|$.

Proposition 5. *Let G be a graph, ϕ be an automorphism of G with order 2 and P a permutation matrix of ϕ. If an eigenvalue of $\mathcal{L}(G)$ is simple then the corresponding eigenvector is odd or even with respect to ϕ.*

Proof. Let λ be an eigenvalue, U an eigenvector of $\mathcal{L}(G)$. If λ is simple then PU and U are linearly dependent. Then there exists a constant c such that $PU = cU$. Since $P^2 = I$ for an automorphism of order 2, $IU = cPU = c^2U$ and $c = \pm 1$. Then $PU = U$ or $PU = -U$. Hence an eigenvector U is odd or even with respect to ϕ. □

Definition 25. Let $G = (V, E)$ be a graph, $V = \{v_i \mid 1 \leq i \leq n\}$ $(n = |V|)$ and $U = (u_1, u_2, \ldots, u_n) \in \Re^n$ a vector. We define three subsets of V as follows:

$$\begin{aligned} V^+(U) &= \{v_i \in V \mid u_i > 0\}, \\ V^-(U) &= \{v_i \in V \mid u_i < 0\}, \textit{ and} \\ V^0(U) &= \{v_i \in V \mid u_i = 0\}. \end{aligned}$$

Lemma 4. *Let $\mathcal{L}(G)$ be the normalized Laplacian of graph G and $U = (u_i), (i = 1, \ldots, n)$ the second eigenvector. If $U \neq \mathbf{0}$ then $V^+(U) \neq \emptyset$ and $V^-(U) \neq \emptyset$.*

Proof. The vector $D^{1/2}\vec{1}$ is an eigenvector corresponding to the zero eigenvalue. Since the second eigenvector U is orthogonal to $D^{\frac{1}{2}}\vec{1}$, $(D^{\frac{1}{2}}\vec{1})^TU = 0$ and $\sum_i \sqrt{d_i}u_i = 0$. Since $d_i > 0, U \neq \mathbf{0}$, there exist at least two values such that $u_i > 0$ and $u_j < 0$ for $i \neq j$. Hence $V^+(U) \neq \emptyset$ and $V^-(U) \neq \emptyset$. □

Lemma 5. *Let G be a graph with an automorphism ϕ of order 2. Let $U = (u_1, u_2, \ldots, u_n)$ be an eigenvector and $\phi(U) = (u_{\phi(1)}, u_{\phi(2)}, \ldots, u_{\phi(n)})$. If $U \neq \mathbf{0}$ and $\phi(U) = -U$ then $V^+(U) \neq \emptyset$ and $V^-(U) \neq \emptyset$.*

Proof. Assume $V^+(U) = \emptyset$. If $u_i \leq 0, (i = 1, \ldots, n)$, $\phi(U) = -U$ implies that $u_{\phi(i)} > 0$. This contradicts that $V^+(U) = \emptyset$. Similarly, if we assume that $V^-(U) = \emptyset$ and $u_i \geq 0$ for $(i = 1, \ldots, n)$, then $\phi(U) = -U$ implies that $u_{\phi(i)} < 0$. Then this contradicts that $V^-(U) = \emptyset$. If $u_i = 0, (i = 1, \ldots, n)$, then $U = \mathbf{0}$ and contradicts that $U \neq \mathbf{0}$. □

Proposition 6 (Guattery et al.[9])**.** *Let P_n be a weighted path graph and $\mathcal{L}(P_n)$ be its normalized Laplacian matrix. For any eigenvector $X = (x_1, x_2, \ldots, x_n)$,*

1. $x_1 = 0$ *implies* $X = 0$,

2. $x_n = 0$ *implies* $X = 0$ *and,*

3. $x_i = x_{i+1} = 0$ *implies that* $X = 0$.

□

Lemma 6 (Guattery et al.[9]). *For a path graph P_n, $\mathcal{L}(P_n)$ has n simple eigenvalues.*

Proof. Let $U = (u_1, u_2, \ldots, u_n)$ and $\bar{U} = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n)$ be two eigenvectors of $\mathcal{L}(P_n)$ with eigenvalue λ. From the proposition 6, we have $u_n \neq 0$ and $\bar{u}_n \neq 0$. Let $\alpha = \dfrac{\bar{u}_n}{u_n}$, where $\alpha \neq 0$. Consider $\mathcal{L}(P_n)(\alpha U - \bar{U}) = \lambda(\alpha U - \bar{U})$. The n-th element of $(\alpha U - \bar{U})$ is $(\bar{u}_n u_n - \bar{u}_n u_n) = 0$. Then $\alpha U = \bar{U}$. Thus U and $\bar{U}$ are linearly dependent and hence λ is simple. □

Proposition 7. *Let P_n be the path graph and ϕ the automorphism of order 2 defined on $V(P_n)$. Then any second eigenvector U_2 of $\mathcal{L}(P_n)$ is an odd vector.*

□

Example 2. Let

$$M = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

and

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

If U_M is a second eigenvector of M then by the Proposition 4, PU_M is also a second eigenvector. By the Proposition 5, $PU_M = U_M$ or $PU_M = -U_M$. By the Proposition 7, U_M is an odd vector and $PU_M = -U_M$.

Definition 26 (Weighted Path). For n $(n \geq 1)$ and k $(k \geq 1)$, the adjacency matrix (P_{ij}) of a weighted path $P_{n,k} = (V, E)$ is the $(n+k) \times (n+k)$ matrix such that

$$P_{ij} = \begin{cases} 0 & (i = j \text{ and } i \leq n) \text{ or } (i \neq j+1 \text{ and } j \neq i+1), \\ 1 & (i = j \text{ and } n+1 \leq i) \text{ or } (i = j+1 \text{ or } j = i+1). \end{cases}$$

That is $V = \{x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+k}\}$ and $E = \{(x_i, x_j) \mid P_{ij} = 1, 1 \leq i, j \leq n+k\}$.

Let Σ be an alphabet and Σ^* a set of strings over Σ including the empty string ϵ. We denote the length of $w \in \Sigma^*$ by $|w|$. Let $\Sigma^{<n} = \{w \in \Sigma^* \| w| < n\}$ and $\Sigma_1^{<n} = \{w \in \Sigma^* | 1 \leq |w| < n\}$. In this paper, we assume $\Sigma = \{0, 1\}$.

Definition 27 (Complete binary tree). A complete binary tree $T_n = (V, E)$ of depth n is defined as follows.

$$\begin{aligned} V &= \Sigma^{<n}, \\ E &= \{(w, wu) \mid w \in \Sigma^{<(n-1)}, u \in \Sigma\}. \end{aligned}$$

Definition 28 (Double tree). A double tree $DT_n = (V, E)$, where n is the depth of the tree, consists of two complete binary trees connected by their root. We define double tree as follows.

$$\begin{aligned} V &= \{x(w) \mid w \in \Sigma^{<n}\} \cup \{y(w) \mid w \in \Sigma^{<n}\}, \\ E_1 &= \{(x(w), x(wu)) \mid w \in \Sigma^{<(n-1)}, u \in \Sigma\}, \\ E_2 &= \{(y(w), y(wu)) \mid w \in \Sigma^{<(n-1)}, u \in \Sigma\}, \\ E &= E_1 \cup E_2 \cup \{(x(\epsilon), y(\epsilon))\}. \end{aligned}$$

Definition 29 (Cycle). A cycle $C_n = (V_n, E_n)$ consists of a vertex set $V_n = \{v_l \mid l \in \mathbb{Z}^+, l \leq n\}$ and an edge set $E_n = \{(v_l, v_{l+1}) \mid 1 \leq l < n\} \cup \{(v_1, v_n)\}$.

Definition 30 (Complete graph). A complete graph $K_n = (V_n, E_n)$ consists of a vertex set $V_n = \{v_i \mid 1 \leq i \leq n\}$ and an edge set $E_n = \{(v_i, v_j) \mid i \neq j \text{ and } 1 \leq i \leq n, 1 \leq j \leq n\}$.

Definition 31 (Graph $R_{n,k}$). The graph $R_{n,k} (n \geq 1, k \geq 2)$ is a bounded degree planer graph with a vertex set $V = V_1 \cup V_2$ and an edge set $E = E_1 \cup E_2 \cup E_3$.

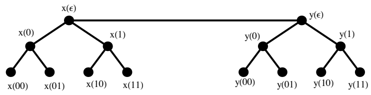
$$\begin{aligned} V_1 &= \{x_i \mid 1 \leq i \leq n+k\}, \\ V_2 &= \{y_i \mid 1 \leq i \leq n+k\}, \\ E_1 &= \{(x_i, x_{i+1}) \mid 1 \leq i \leq n+k-1\}, \\ E_2 &= \{(y_i, y_{i+1}) \mid n+k+1 \leq i \leq 2(n+k)-1\}, \\ E_3 &= \{(x_i, y_i) \mid n+1 \leq i \leq n+k\}. \end{aligned}$$

Definition 32 (Cycle cross paths $C_m \square P_n$). Let C_m be a cycle with $V = \{c_i \mid 1 \leq i \leq m\}$ and $E = \{(c_i, c_{i+1}) \mid 1 \leq i < m\} \cup \{(c_1, c_m)\}$. Let P_n be a path with $V = \{p_i \mid 1 \leq i \leq n\}$ and $E = \{(p_i, p_{i+1}) \mid 1 \leq i < n\}$. Graph $C_m \square P_n$ has n copies of cycles C_m, each corresponding to the one vertex of the path graph. A vertex set V and an edge set $E = E_1 \cup E_2 \cup E_3$ of $C_m \square P_n$ is defined as follows.

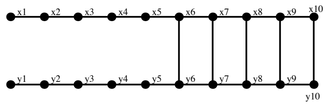
$$\begin{aligned} V &= \{(c_i, p_j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}, \\ E_1 &= \bigcup_{i=1}^{m} \{((c_i, p_j), (c_i, p_{j+1})) \mid 1 \leq j \leq n-1\}, \\ E_2 &= \bigcup_{j=1}^{n} \{((c_i, p_j), (c_{i+1}, p_j)) \mid 1 \leq i \leq m-1\}, \\ E_3 &= \{((c_1, p_i), (c_m, p_i)) \mid 1 \leq i \leq n\}, \\ E &= E_1 \cup E_2 \cup E_3. \end{aligned}$$

Example 3. Double tree DT_3 shown in the Figure 1(a) has a vertex set $V = \{x(\epsilon), x(0), x(1), y(\epsilon), y(0), y(1), x(00), x(01), x(10), x(11), y(00), y(01), y(10), y(11)\}$ and an edge set $E = \{(x(\epsilon), y(\epsilon)), (x(\epsilon), x(0)), (x(\epsilon), x(1)), (y(\epsilon), y(0)), (y(\epsilon), y(1)), (x(0), x(00)), (x(0), x(01)), (x(1), x(10)), (x(1), x(11)), y(0), y(00)), ((y(0), y(01)), (y(1), y(10)), (y(1), y(11))$. Graph $R_{5,5}$ shown in the Figure 1(b) has a vertex set $V = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\}$ and an edge set $E = \{(x_6, y_6), (x_7, y_7), (x_8, y_8), (x_9, y_9), (x_{10}, y_{10})\}$.

Definition 33 (Lollipop graph $LP_{n,m}$). The lollipop graph $LP_{n,m}, (n \geq 3, m \geq 1)$ is obtained by connecting a vertex of K_n to the end vertex of P_m as shown in the Figure 2. We start vertex numbering from the end vertex of the path. Define



(a) Double Tree DT_3



(b) $R_{5,5}$

Figure 1: Double tree DT_3 and graph $R_{n,k}(n = 5, k = 5)$.

$LP_{n,m} = (V, E)$ as follows.

$$
\begin{aligned}
V &= \{x_1, x_2, \ldots, x_m, y_1, \ldots, y_n\},\\
E &= \{(x_i, x_{i+1}) \mid 1 \le i \le m-1\} \cup \{(y_i, y_j) \mid i \ne j,\\
&\quad 1 \le i \le n, 1 \le j \le n\} \cup \{(x_m, y_1)\}.
\end{aligned}
$$

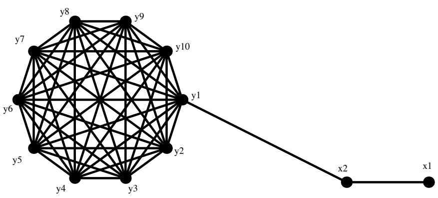


Figure 2: Graph $LP_{n,m}(n = 10, m = 2)$

3. Minimum normalized cut of graphs

We use the term $Mcut(G)$ to represent the minimum normalized cut. In this section, we review the basic properties of $Mcut(G)$ and its relation to the connectivity and second smallest eigenvalue of normalized Laplacian. We derive $Mcut(G)$ of basic classes of graphs such as paths, cycles, double trees, cycle cross paths, complete graphs and other graphs such as $R_{n,k}$, $P_{n,k}$ and $LP_{n,m}$.

3.1. Properties of minimum normalized cut $Mcut(G)$

Definition 34 (Normalized cut)**.** Let $G = (V, E)$ be a connected graph. Let $A, B \subset V$, $A \ne \emptyset$, $B \ne \emptyset$ and $A \cap B = \emptyset$. Then the normalized cut $Ncut(A, B)$ of G is defined by

$$Ncut(A, B) = cut(A, B)\left(\frac{1}{vol(A)} + \frac{1}{vol(B)}\right).$$

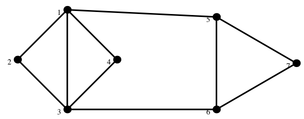
Definition 35 ($Mcut(G)$)**.** Let $G = (V, E)$ be a connected graph. The $Mcut(G)$ is defined by

$$Mcut(G) = \min\{Mcut_j(G) \mid j = 1, 2, \ldots\}.$$

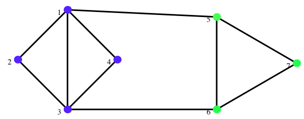
Where,

$$Mcut_j(G) = \min\{Ncut(A, V \setminus A) \mid cut(A, V \setminus A) = j, A \subset V\}.$$

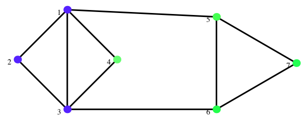
Example 4. Graph $G = (V, E)$ shown in the Figure 3 has vertex set $V = \{1, 2, 3, 4, 5, 6, 7\}$ and edge set $E = \{(1, 2), (2, 3), (3, 1), (3, 4), (1, 4), (1, 5), (3, 6), (6, 5), (7, 5), (7, 6)\}$. Volume of the graph is 20. We compute normalized cut for the following cases.
$Case(1)$ $A = \{1, 2, 3, 4\}, B = \{5, 6, 7\}, vol(A) = 12, vol(B) = 8$, $cut(A, B) = 2$ and $Ncut(A, B) = 0.417$.
$Case(2)$ $A = \{1, 2, 3\}, B = \{4, 5, 6, 7\}, vol(A) = 10, vol(B) = 10$, $cut(A, B) = 4$ and $Ncut(A, B) = 0.8$.
$Case(3)$ $A = \{1, 3, 4, 5, 6, 7\}, B = \{2\}, vol(A) = 2, vol(B) = 18$, $cut(A, B) = 2$ and $Ncut(A, B) = 1.1111$. Comparing above 3 cases, we obtain $Mcut(G)$ for the case(1).



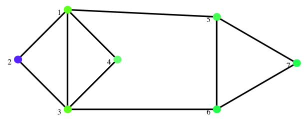
(a) $G = (V, E)$



(b) $Case(1)$



(c) $Case(2)$



(d) $Case(3)$

Figure 3: Normalized cut example.

Lemma 7. *Let $G = (V, E)$ be a connected graph. Then $\left(\frac{1}{vol(A)} + \frac{1}{vol(V \setminus A)}\right)$ is minimum when $vol(A) = vol(V \setminus A) = \frac{vol(G)}{2}$.*

□

Proposition 8. *Let $G = (V, E)$ be a connected graph, $A \subseteq V$ and $\triangle(G)$ the maximum degree of G. Then*

1. *$cut(A, V \setminus A) \ge \kappa'(G)$,*
2. *$Mcut(G) \ge \dfrac{4\kappa'(G)}{\triangle(G)|V|}$ and*
3. *If $cut(A, V \setminus A) = \kappa'(G)$ and $2vol(A) = vol(G)$ then $Mcut(G) = \dfrac{4\kappa'(G)}{vol(G)}$.*

Proof.

1. Since $\kappa'(G)$ is the edge connectivity, $cut(A, V\setminus A) \geq \kappa'(G)$ for any $A \subseteq V$.

2. From Lemma 7, $\left(\frac{1}{vol(A)} + \frac{1}{vol(V\setminus A)}\right)$ is minimum when $vol(A) = vol(V\setminus A)$. That is $\left(\frac{1}{vol(A)} + \frac{1}{vol(V\setminus A)}\right) \geq \frac{2}{vol(A)} = \frac{4}{vol(G)}$. Since $vol(G) = \sum_{i=1}^{|V|} d_i \leq |V|\triangle(G)$, $Ncut(A, V\setminus A) = cut(A, V\setminus A)\left(\frac{1}{vol(A)} + \frac{1}{vol(V\setminus A)}\right) \geq \frac{4\kappa'(G)}{\triangle(G)|V|}$.

3. If $cut(A, V\setminus A) = \kappa'(G)$ and $2vol(A) = vol(G)$ then it is clear that, $Mcut(G) = \frac{4\kappa'(G)}{vol(G)}$.

□

Proposition 9 (Luxburg [16])**.** *Let $G = (V, E)$ be a connected graph and $A \subset V$. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be eigenvalues of $\mathcal{L}(G)$. Then $Mcut(G) \geq \lambda_2(\mathcal{L}(G))$.*

Proof. Let $V = \{1, 2, \ldots, n\}$. Let $A \subset V$, $g = (g_1, \ldots, g_n) \in \Re^n$ an eigenvector and $g = D^{1/2} f$. Define f_i as

$$f_i = \begin{cases} a & \text{if } i \in A, \\ -b & \text{if } i \notin A. \end{cases}$$

Then

$$\frac{\sum_{i=1}^n \sum_{j=1}^n (f_i - f_j)^2 w_{ij}}{2\sum_{i=1}^n f_i^2 d_i} = \frac{2cut(A, (V\setminus A))(a+b)^2}{2(a^2 vol(A) + b^2 vol(V\setminus A))}.$$

Let this as X.

Now let $a = vol(V\setminus A)$ and $b = vol(A)$. Then we have

$$\begin{aligned} X &= \frac{cut(A, (V\setminus A))(vol(G))^2}{vol(V\setminus A)^2 vol(A) + vol(A)^2 vol(V\setminus A)} \\ &= \frac{cut(A, (V\setminus A))(vol(G))^2}{vol(V\setminus A)vol(A)(vol(A) + vol(V\setminus A)} \\ &= \frac{cut(A, (V\setminus A))(vol(G))}{vol(V\setminus A)vol(A)} \\ &= cut(A, (V\setminus A))\left(\frac{1}{vol(V\setminus A)} + \frac{1}{vol(A)}\right) \\ &= Ncut(A, V\setminus A). \end{aligned}$$

With the choice of f, a, b we have, $(D\vec{1})^T f = \sum_{i=1}^n d_i f_i = \sum_{i\in A} d_i a - \sum_{i\notin A} d_i b = 0$. So $f \perp D\vec{1}$. Since $\lambda_2 = \inf_{f\perp D\vec{1}} \frac{\sum_{i=1}^n \sum_{j=1}^n (f_i - f_j)^2 w_{ij}}{2\sum_{i=1}^n f_i^2 d_i}$, we have $\lambda_2 \leq \frac{\sum_{i=1}^n \sum_{j=1}^n (f_i - f_j)^2 w_{ij}}{2\sum_{i=1}^n f_i^2 d_i} = \min_A Ncut(A, (V\setminus A)) = Mcut(G)$. □

Lemma 8. *Let $G = (V, E)$ be a connected graph, A a nonempty subset of V. Then*

(i) $Ncut(A, V\setminus A) = \frac{4cut(A, V\setminus A)\cdot vol(V)}{(vol(V))^2 - (vol(A) - vol(V\setminus A))^2}$, *and*

(ii) $Mcut_j(G) = \frac{4j vol(V)}{vol(V)^2 - X_j}$, *where* $X_j = \min\{(vol(A) - vol(V\setminus A))^2 \mid cut(A, V\setminus A) = j,\ A \subset V\}$.

Proof. (i) Let $s = vol(V)$, $j = cut(A, V\setminus A)$, $s_A = vol(A)$ and $s_{\bar{A}} = vol(V\setminus A)$. Since $s = s_A + s_{\bar{A}}$, we have $s_A - s_{\bar{A}} = 2s_A - s$ and $s^2 - (s_A - s_{\bar{A}})^2 = 4s_A s_{\bar{A}}$.

$$\begin{aligned} Ncut(A, V\setminus A) &= j\cdot\left(\frac{1}{s_A} + \frac{1}{s_{\bar{A}}}\right) \\ &= \frac{j(s_A + s_{\bar{A}})}{s_A s_{\bar{A}}} = \frac{js}{s_A s_{\bar{A}}} \\ &= \frac{4js}{s^2 - (s_A - s_{\bar{A}})^2} \end{aligned}$$

(ii) It is followed by the definition of $Mcut_j(G)$ and (i). □

Lemma 9. *Let $G = (V, E)$ be a graph. If there exists a nonempty subset $A \subset V$ such that*

$$|vol(A) - vol(V\setminus A)| \leq \frac{vol(V)}{\sqrt{cut(A, V\setminus A) + 1}},$$

then

$$Mcut(G) = \min\{Mcut_j(G) \mid j = 1, 2, \ldots, cut(A, V\setminus A)\}.$$

Proof. Let $j = cut(A, V\setminus A)$, $a = |vol(A) - vol(V\setminus A)|$, $s = vol(V)$, $s_A = vol(A)$ and $s_{\bar{A}} = vol(V\setminus A)$. Since $a^2 \leq \frac{s^2}{j+1}$ and $Ncut(A, V\setminus A) = \frac{4js}{s^2 - a^2}$ by the Lemma 8, we have $s^2 - (j+1)a^2 \geq 0$ and

$$\begin{aligned} &\frac{4(j+1)}{s} - Ncut(A, V\setminus A) \\ &= \frac{4(j+1)}{s} - \frac{4js}{s^2 - a^2} \\ &= \frac{4(j+1)(s^2 - a^2) - 4js^2}{s(s^2 - a^2)} \\ &= \frac{4(s^2 - (j+1)a^2)}{s(s^2 - a^2)} \geq 0. \end{aligned}$$

Let B be a subset of V, $s_B = vol(B)$, $s_{\bar{B}} = vol(V\setminus B)$ and $j_B = cut(B, V\setminus B)$. If $j_B \geq j+1$ then we have the following using Lemma 8.

$$\begin{aligned} Ncut(B, V\setminus B) &= cut(B, V\setminus B)\left(\frac{1}{vol(B)} + \frac{1}{vol(V\setminus B)}\right) \\ &= \frac{4j_B s}{s^2 - (s_B - s_{\bar{B}})^2} \\ &\geq \frac{4(j+1)s}{s^2 - (s_B - s_{\bar{B}})^2} \\ &\geq \frac{4(j+1)s}{s^2} = \frac{4(j+1)}{s} \\ &\geq Ncut(A, V\setminus A) \geq Mcut_j(G). \end{aligned}$$

So we have $Mcut_{j'}(G) \geq Mcut_j(G)$ for any $j' > j$. □

Lemma 10. *Let $G = (V, E)$ be a graph with $vol(G) \geq 9$. If there exists a subset $A \subset V$ such that $cut(A, V\setminus A) = 1$ and $|vol(A) - vol(G)/2| \leq 3$, then*

$$Mcut(G) = Mcut_1(G).$$

Proof. Let $s = vol(G)$, $s_A = vol(A)$ and $s_{\bar{A}} = vol(V \setminus A)$. Since $\left|s_A - \frac{s}{2}\right| \leq 3$ and $s = s_A + s_{\bar{A}}$, we have $|s_A - s_{\bar{A}}| \leq 6$. Since $\sqrt{1+1}\,|s_A - s_{\bar{A}}| \leq 6\sqrt{2} < 9 \leq s$, we have $Mcut(G) = Mcut_1(G)$ by the Lemma 9. □

Lemma 11. *Let $G = (V, E)$ be a graph and $vol(G) \geq 11$. If there exists a set $A \subset V$ such that $cut(A, V \setminus A) = 2$ and $|vol(A) - vol(G)/2| \leq 3$, then*

$$Mcut(G) = \min(Mcut_1(G), Mcut_2(G)).$$

Proof. Since $|vol(A) - vol(G)/2| \leq 3$ and $vol(A) + vol(V \setminus A) = vol(G)$, we have $|vol(A) - vol(V \setminus A)| \leq 6$ and $\sqrt{3}\,|vol(A) - vol(V \setminus A)| \leq 6\sqrt{3} < 11$. So we have $Mcut(G) = \min(Mcut_1(G), Mcut_2(G))$ by the Lemma 9. □

Lemma 12. *Let $G = (V, E)$ be a graph with $vol(G) \geq 11$. Suppose there exists a subset $A \subset V$ such that $cut(A, V \setminus A) = 2$ and $|vol(A) - vol(G)/2| \leq 3$. If there exists no subset $B \subset V$ such that $cut(B, V \setminus B) = 1$ and $|vol(B) - vol(G)/2| \leq \dfrac{\sqrt{36 + (vol(G))^2}}{2\sqrt{2}}$, then*

$$Mcut(G) = Mcut_2(G).$$

Proof. Let $s = vol(G)$, $s_A = vol(A)$ and $s_{\bar{A}} = vol(V \setminus A)$. Since $|s_A - s/2| \leq 3$, we have $|s_A - s_{\bar{A}}| \leq 6$ and

$$\begin{aligned} Mcut_2(G) &\leq Ncut(A, V \setminus A) \\ &= \frac{8s}{s^2 - (s_A - s_{\bar{A}})^2} \\ &\leq \frac{8s}{s^2 - 36}. \end{aligned}$$

Let $B \subset V$ with $cut(B, V \setminus B) = 1$, $s_B = vol(B)$ and $s_{\bar{B}} = vol(V \setminus B)$. If B exists, then $|s_B - s/2| > \frac{\sqrt{s^2+36}}{2\sqrt{2}}$, by the assumption. So we have $|s_B - s_{\bar{B}}| > \frac{\sqrt{s^2+36}}{\sqrt{2}}$ and

$$\begin{aligned} Ncut(B, V \setminus B) &= \frac{4s}{s^2 - (s_B - s_{\bar{B}})^2} \\ &\geq \frac{4s}{s^2 - \frac{s^2+36}{2}} \\ &= \frac{8s}{s^2 - 36} \geq Mcut_2(G). \end{aligned}$$

That is $Mcut(G) = Mcut_2(G)$ by the Lemma 11. □

Next we derive formulae for minimum normalized cut $Mcut(G)$ of some elementary graphs.

3.2. $Mcut(G)$ of basic classes of graphs

Theorem 1. *Let $G = (V, E)$ be a graph.*

1. *If G is a regular graph of degree d and $G \neq K_n, n > 3$ and $|V| = n$, then*

$$Mcut(G) \geq \begin{cases} \frac{4}{n} & \text{if } n \text{ is even,} \\ \frac{4n}{(n^2-1)} & \text{if } n \text{ is odd.} \end{cases}$$

2. *For the cycle C_n ($n \geq 3$),*

$$Mcut(C_n) = \begin{cases} \frac{4}{n} & \text{if } n \text{ is even,} \\ \frac{4n}{(n^2-1)} & \text{if } n \text{ is odd.} \end{cases}$$

This can be written as $Mcut(C_n) = \dfrac{n}{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil}$.

3. *For the complete graph K_n,*

$$\begin{aligned} Mcut(K_n) &= \frac{n}{n-1} \\ &= \lambda_2. \end{aligned}$$

4. *For the path graph P_n ($n \geq 2$),*

$$Mcut(P_n) = \begin{cases} \frac{2}{n-1} & \text{if } n \text{ is even,} \\ \frac{2(n-1)}{n(n-2)} & \text{if } n \text{ is odd.} \end{cases}$$

This can be written as

$$Mcut(P_n) = \frac{2n-2}{4\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil - 2n + 1}.$$

5. *For the cycle cross paths $G = C_m \Box P_n$,*

$$Mcut(C_m \Box P_n) = \begin{cases} \frac{2(2n-1)}{16\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil - 4n + 1} & 2n > m, \\ \frac{nm}{(2n-1)\lfloor \frac{m}{2} \rfloor \lceil \frac{m}{2} \rceil} & 2n \leq m. \end{cases}$$

6. *For the double tree DT_n with depth n, $Mcut(DT_n) = \dfrac{2}{2^{n+1} - 3}$.*

Proof. 1. For a regular graph of degree d, $\kappa'(G) = \triangle(G) = \delta(G) = d$. For $A \subset V$, $Ncut(A, V \setminus A) \geq \kappa'(G)\left(\dfrac{1}{d|A|} + \dfrac{1}{d|V \setminus A|}\right) = \dfrac{|V|}{|A||V \setminus A|}$. If $cut(A, V \setminus A) = \kappa'(G)$ then we have $Ncut(A, V \setminus A) = \dfrac{|V|}{|A||V \setminus A|}$. $Ncut(A, V \setminus A)$ is minimum, when $|A| = |V \setminus A|$ by the Lemma 7. If V is even then $Mcut(G) \geq \dfrac{4}{|V|} = \dfrac{4}{n}$ by Lemma 7.

If $|V|$ is odd then, we can write $|V|$ as $|V| = \dfrac{|V|-1}{2} + \dfrac{|V|+1}{2}$, where $-1 \leq |A| - |V \setminus A| \leq 1$. Then $Ncut(A, V \setminus A) \geq \kappa'(G)\left(\dfrac{2}{d(|V|-1)} + \dfrac{2}{d(|V|+1)}\right) = \dfrac{4|V|}{(|V|+1)(|V|-1)}$. Hence $Mcut(G) \geq \dfrac{4|V|}{(|V|+1)(|V|-1)} = \dfrac{4n}{n^2-1}$. □

2. Let $A_k = \{x_i \mid i \leq k\}$ ($k = 1, \ldots, n-1$). We note $vol(C_n) = 2n$, $vol(A_k) = 2k$, $vol(V \setminus A_k) = 2n - 2k$, $vol(A_k) - vol(V \setminus A_k) = 4k - 2n$ and

$$Ncut(A_k, V \setminus A_k) = \frac{4n}{n^2 - (2k-n)^2}.$$

If n is even then $Ncut(A_{\frac{n}{2}}, V \setminus A_{\frac{n}{2}}) = \frac{4}{n}$ is the minimum of $Ncut(A_k, V \setminus A_k)$. If n is odd then $Ncut(A_{\frac{n+1}{2}}, V \setminus A_{\frac{n+1}{2}}) = Ncut(A_{\frac{n-1}{2}}, V \setminus A_{\frac{n-1}{2}}) = \frac{4n}{n^2-1}$ is the minimum of $Ncut(A_k, V \setminus A_k)$. Since $vol(A_{\frac{n}{2}}) - vol(V \setminus A_{\frac{n}{2}}) = 0$, $vol(A_{\frac{n-1}{2}}) - vol(V \setminus$

$A_{\frac{n-1}{2}}) = -2$ and $\dfrac{vol(V)}{\sqrt{cut(A_k, V \setminus A_k) + 1}} = \frac{2n}{\sqrt{3}} \geq \frac{6}{\sqrt{3}}$, we have $Mcut(C_n) = Mcut_2(C_n)$ by Lemma 9.

We note that for any nonempty subset $A \subset V$ with $cut(A, V \setminus A) = 2$, there exists a k such that $Ncut(A, V \setminus A) = Ncut(A_k, V \setminus A_k)$ and $\kappa'(C_n) = 2$.

For even n, $n = \lfloor \dfrac{n}{2} \rfloor = \lceil \dfrac{n}{2} \rceil$ and for odd n, $\dfrac{(n-1)}{2} = \lfloor \dfrac{n}{2} \rfloor$ and $\dfrac{(n+1)}{2} = \lceil \dfrac{n}{2} \rceil$. Combining odd and even cases together we can write $Mcut(C_n)$ as $Mcut(C_n) = \dfrac{4n}{4\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil}$. □

3. For a complete graph K_n, $|V| = n$, $\kappa'(K_n) = n - 1$ and $vol(K_n) = n(n-1)$. For any subset $A \subset V$, we have $vol(A) = |A|(n-1)$ and $cut(A, (V \setminus A)) = |A|(n-|A|)$. Then $Mcut(K_n) = |A|(n-|A|)\left(\dfrac{1}{|A|(n-1)} + \dfrac{1}{(n-|A|)(n-1)}\right) = \dfrac{n}{n-1}$. □

4. Let $A_k = \{x_i \mid i \leq k\}$ $(k = 1, \ldots, n-1)$. We note that $vol(P_n) = 2n-2$, $vol(A_k) = 2k-1$, $vol(V \setminus A_k) = 2n-2k-1$, $vol(A_k) - vol(V \setminus A_k) = 4k - 2n$ and
$$Ncut(A_k, V \setminus A_k) = \frac{2(n-1)}{(n-1)^2 - (2k-n)^2}.$$
If n is even then $Ncut(A_{\frac{n}{2}}, V \setminus A_{\frac{n}{2}}) = \frac{2}{n-1}$ is the minimum of $Ncut(A_k, V \setminus A_k)$. If n is odd then $Ncut(A_{\frac{n+1}{2}}, V \setminus A_{\frac{n+1}{2}}) = Ncut(A_{\frac{n-1}{2}}, V \setminus A_{\frac{n-1}{2}}) = \frac{2(n-1)}{(n-1)^2-1} = \frac{2(n-1)}{n(n-2)}$ is the minimum of $Ncut(A_k, V \setminus A_k)$. Since $vol(A_{\frac{n}{2}}) - vol(V \setminus A_{\frac{n}{2}}) = 0$, $vol(A_{\frac{n-1}{2}}) - vol(V \setminus A_{\frac{n-1}{2}}) = -2$ and $\dfrac{vol(V)}{\sqrt{cut(A_k, V \setminus A_k) + 1}} = \frac{2n-2}{\sqrt{2}} \geq \frac{2}{\sqrt{2}}$, we have $Mcut(C_n) = Mcut_1(P_n)$ by Lemma 9. □

5. The cycle cross path $G = C_m \square P_n (n \geq 2, m \geq 3)$ is a graph which has n copies of cycles C_m, each corresponding to the one vertex of P_n. $\kappa'(C_m \square P_n) = \min\{\kappa'(C_m)|V(P_n)|, \kappa'(P_n)|V(C_m)|, \delta(C_m) + \delta(P_n)\} = \delta(C_m) + \delta(P_n) = 3$.
Case (i) Let $A_1 = \{(c_i, p_j) | 1 \leq j \leq \lfloor \dfrac{n}{2} \rfloor, 1 \leq i \leq m\}$ and $V \setminus A_1 = \{(c_i, p_j) | \lfloor \dfrac{n}{2} \rfloor + 1 \leq j \leq n, 1 \leq i \leq m\}$. We note that $vol(A_1) = \lfloor \dfrac{n}{2} \rfloor (vol(C_m) + 2m) - m$, $vol(V \setminus A_1) = \lceil \dfrac{n}{2} \rceil (vol(C_m) + 2m) - m$ and $cut(A_1, V \setminus A_1) = m$. Then $Ncut(A_1, V \setminus A_1) = \dfrac{m(4mn - 2m)}{(\lfloor \frac{n}{2} \rfloor 4m - m)(\lceil \frac{n}{2} \rceil 4m - m)} = \dfrac{2(2n-1)}{16\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil - 4n + 1}$. When n is even, $Ncut(A_1, V \setminus A_1) = \dfrac{2}{2n-1}$. When n is odd, $Ncut(A_1, V \setminus A_1) = \dfrac{2(2n-1)}{(2n-3)(2n+1)}$.
Case (ii) Let $A_2 = \{(c_i, p_j) \mid 1 \leq i \leq \lfloor \dfrac{m}{2} \rfloor, 1 \leq j \leq n\}$ and $(V \setminus A_2) = \{(c_i, p_j) \mid \lfloor \dfrac{m}{2} \rfloor + 1 \leq i \leq m, 1 \leq j \leq n\}$. We note that $vol(A_2) = n vol(C_{\lfloor \frac{m}{2} \rfloor}) + 2\lfloor \dfrac{m}{2} \rfloor (n-1) = 2(2n-1)\lfloor \dfrac{m}{2} \rfloor$ and $vol(V \setminus A_2) = 2(2n-1)\lceil \dfrac{m}{2} \rceil$. In this case, the graph cut horizontally through the cycles and we have $cut(A_2, V \setminus A_2) = 2n$. Hence $Ncut(A_2, V \setminus A_2) = \dfrac{nm}{(2n-1)\lfloor \frac{m}{2} \rfloor \lceil \frac{m}{2} \rceil}$. When m is odd, $\dfrac{4nm}{(2n-1)(m^2-1)}$ and when m is even, $\dfrac{4n}{(2n-1)m}$.

Case (iii) Let $B_k = \{(c_i, p_1) | 1 \leq i \leq k\}$ $(1 \leq k < m)$. We note that $cut(B_k, V \setminus B_k) = k + 2$ and $vol(B_k) = 3k$. Since $Ncut(A_1, V \setminus A_1) - Ncut(B_k, V \setminus B_k) = \frac{2(-1+2n)\left(9k^2+km\left(3-16n+4n^2\right)+2m\left(-3-4n+4n^2\right)\right)}{3k(3k+m(2-4n))\left(-3-4n+4n^2\right)}$, we can verify $Ncut(A_1, V \setminus A_1) \leq Ncut(B_k, V \setminus B_k)$ for any k and $m \leq 2n$.

Case (iv) Let $C_k = \{(c_1, p_j) | 1 \leq j \leq k\}$ $(1 \leq k < n)$. We note that $cut(C_k, V \setminus B_k) = 2k + 1$ and $vol(C_k) = 4k - 1$. Since $Ncut(C_k, V \setminus C_k) = -\frac{2(1+2k)m(-1+2n)}{(-1+4k)(-1+4k+m(2-4n))}$, we can verify $Ncut(A_2, V \setminus A_2) \leq Ncut(C_k, V \setminus C_k)$ for any k and $2n \leq m$.

Now compare the case (i) with case (ii).

For the case of $2n \geq m + 1$, we have $\frac{vol(G)}{\sqrt{cut(A_1, V \setminus A_1)}} = \frac{2m(2n-1)}{\sqrt{m+1}} \geq \frac{2m(2n-1)}{\sqrt{2n}} = 2m\sqrt{\frac{4n^2-4n+1}{2n}} \geq 2m\sqrt{2n-2} \geq 4m \geq |vol(A_1) - vol(V \setminus A_1)|$ and $Mcut_{2n}(G) > Mcut_m(G)$. So $Mcut(G) = Ncut(A_1, V \setminus A_1)$.

If $2n \leq m$, then we have $\frac{vol(G)}{\sqrt{cut(A_2, V \setminus A_2)}} = \frac{2m(2n-1)}{\sqrt{2n+1}} \geq \frac{4n(2n-1)}{\sqrt{2n+1}} = 2(2n-1)\sqrt{\frac{4n^2}{1+2n}} = 2(2n-1)\sqrt{2n - \frac{2n}{2n+1}} \geq 2(2n-1)\sqrt{2n-1} \geq 2(2n-1) \geq |vol(A_2) - vol(V \setminus A_2)|$ and $Mcut_m(G) > Mcut_{2n}(G)$. So $Mcut(G) = Ncut(A_2, V \setminus A_2)$.

□

6. The size of a tree is $|T_n| = 1+2+\cdots+2^n = 2^n - 1$ and the size of a double tree is $|DT_n| = 2|T_n| = 2^{n+1} - 2$. The volume of a tree is $vol(T_n) = 2vol(T_{n-1}) + 4$, which can be written as $vol(T_n) + 4 = 2(vol(T_{n-1}) + 4) = 2^2(vol(T_{n-2}) + 4) = \cdots = 2^{n-1}(vol(T_1) + 4) = 2^{n+1}$. Therefore the volume of a tree is $vol(T_n) = 2^{n+1} - 4$ and the volume of a double tree is $vol(DT_n) = 2vol(T_n) + 2 = 2^{n+2} - 6$.

Let $A_1 = \{x(w) \mid w \in \Sigma^{<n}\}$ and $V \setminus A_1 = \{y(w) \mid w \in \Sigma^{<n}\}$. Then we have $vol(A_1) = vol(T_n) + 1 = 2^{n+1} - 3$, $vol(V \setminus A_1) = 2^{n+1} - 3$, $cut(A_1, V \setminus A_1) = 1$.

Therefore $Ncut(A_1, V \setminus A_1) = \frac{2}{(vol(T_n)+1)} = \frac{2}{2^{n+1}-3} = \frac{4}{vol(DT_n)}$.

Here $\kappa'(DT_n) = 1$ and $2vol(A_1) = vol(DT_n)$. Then from the Proposition 7, $Mcut(DT_n) = \dfrac{2}{2^{n+1} - 3}$.

□

3.3. *Mcut* of roach type graphs $R_{n,k}$

Next, we consider the graph $R_{n,k}$ and derive a formula for $Mcut(R_{n,k})$ based on n, k.

Theorem 2. *For $R_{n,k}(n \geq 1, k > 1)$, $Mcut(R_{n,k})$ is given by*

$$\begin{cases} \frac{2}{3} & (n=1, k=2), \\ \frac{4}{-2+3k+2n} & (*_1 \wedge (k \geq 4) \wedge (n < K_1)), \\ \frac{4(-2+3k+2n)}{(-5+3k+2n)(1+3k+2n)} & (*_4 \wedge (k \geq 4) \wedge (n < K_4)), \\ \frac{4(-2+3k+2n)}{(-4+3k+2n)(3k+2n)} & (*_3 \wedge (k \geq 4) \wedge (n < K_3)), \\ \frac{4(-2+3k+2n)}{(-3+3k+2n)(-1+3k+2n)} & (*_2 \wedge (k \geq 4) \wedge (n < K_2)), \vee (n=1, k=3) \vee (n=2, k=3), \\ \frac{6k+4n-4}{(2n-1)(6k+2n-3)} & ((k \geq 4) \wedge ((*_1 \wedge (K_1 \leq n)) \vee (*_4 \wedge (K_4 \leq n)) \vee (*_3 \wedge (K_3 \leq n)) \vee (*_2 \wedge (K_2 \leq n)))) \vee (k=2 \wedge (n \geq 2)) \vee (k=3 \wedge (n \geq 3)), \end{cases}$$

where

$$\begin{aligned} *_1 &= ((3 \mid n) \wedge (2 \mid k)), \\ *_2 &= (3 \nmid n) \textit{ and } (2 \nmid k), \\ *_3 &= (3 \nmid n) \textit{ and } (2 \mid k), \\ *_4 &= ((3 \mid n) \wedge (2 \nmid k)), \\ K_1 &= 1 - \frac{1}{\sqrt{2}} - \frac{3k}{2} + \frac{3k}{\sqrt{2}}, \\ K_2 &= 1 - \frac{3k}{2} + \frac{\sqrt{1-12k+18k^2}}{2}, \\ K_3 &= 1 - \frac{3k}{2} + \frac{\sqrt{-1-6k+9k^2}}{\sqrt{2}}, \\ K_4 &= 1 - \frac{3k}{2} + \frac{\sqrt{-7-12k+18k^2}}{2}. \end{aligned}$$

Proof. Let $V(R_{n,k}) = \{x_i \mid 1 \leq i \leq n+k\} \cup \{y_i \mid 1 \leq i \leq n+k\}$. Volume of $R_{n,k}$ is $vol(R_{n,k}) = 2(2n-1+3k-1) = 6k+4n-4$. We consider the following cases in order to find the $Mcut(R_{n,k})$.

Case(i) Let $A_1 \subseteq V(R_{n,k})$, where $A_1 = \{x_i \mid 1 \leq i \leq n+k\}$ and $V \setminus A_1 = \{y_i \mid 1 \leq i \leq n+k\}$. Then the volume $vol(A_1)$ is $\frac{vol(R_{n,k})}{2} = 2n+3k-2$ and $cut(A_1, V \setminus A_1) = k$. So we have

$$\begin{aligned} Ncut(A_1, V \setminus A_1) &= k\left(\frac{1}{2n+3k-2} + \frac{1}{3k+2n-2}\right) \\ &= \frac{2k}{3k+2n-2}. \end{aligned}$$

Let this value as c_1.

Case(ii) Let $A_2 \subseteq V(R_{n,k})$ such that $A_2 = \{x_i \mid 1 \leq i \leq n\}$ and $V \setminus A_2 = \{x_i \mid n+1 \leq i \leq n+k\} \cup \{y_i \mid 1 \leq i \leq n+k\}$. Then the volume $vol(A_2) = 2n-1$, $vol(V \setminus A_2) = vol(R_{n,k}) - vol(A_2) = 2n+6k-3$ and $cut(A_2, (V \setminus A_2)) = 1$. So we have

$$Ncut(A_2, V \setminus A_2) = \frac{(6k+4n-4)}{(2n-1)(6k+2n-3)}.$$

Let this value as c_2.

Case(iii) Suppose there exists $|A_3| < n$ such that $cut(A_3, (V \setminus A_3)) = 1$. Let $vol(A_3) = 2n-1-2x$, where $x = |A_2| - |A_3|$ and $|A_2| = n$. Then $vol(V \setminus A_3) = 6k+2n-3+2x$. $Ncut(A_3, V \setminus A_3) = \frac{1}{2n-1-2x} + \frac{1}{6k+2n-3+2x} = \frac{6k+4n-4}{(2n-1)(6k+2n-3)+4x(1-(3k+x))}$. Since $4x(1-(3k+x)) < 0$, $Ncut(A_3, V \setminus A_3) > Ncut(A_2, V \setminus A_2)(Case(ii) < Case(iii))$. Since c_2 is smaller than $Case(iii)$, we can ignore this case.

Case(iv) Let $A_4(\alpha) = \{x_i \mid 1 \leq i \leq n+\alpha\} \cup \{y_i \mid 1 \leq i \leq n+\alpha\}$, where $1 \leq \alpha < k$ and $V \setminus A_4(\alpha) = \{x_i \mid n+\alpha+1 \leq i \leq n+k\} \cup \{y_i \mid n+\alpha+1 \leq i \leq n+k\}$. Then $vol(A_4(\alpha)) = 2(2n-1+3\alpha) = 4n+6\alpha-2$, $vol(V \setminus A_4(\alpha)) = 6k-2-6\alpha$ and $cut(A_4(\alpha), V \setminus A_4(\alpha)) = 2$. Then we have,

$$Ncut(A_4(\alpha), V \setminus A_4(\alpha)) = \frac{(3k+2n-2)}{(2n-1+3\alpha)(3k-3\alpha-1)}.$$

Let this value as $c_4(\alpha)$.

Minimum of $c_4(\alpha)$ can be obtained by differentiating with respect to α.

$\frac{dc_4(\alpha)}{d\alpha} = 0$ gives minimum value of $c_4(\alpha)$ at $\alpha_0 = \frac{3k-2n}{6}$. But α_0 is not an integer for all n, k. If $\frac{3k-2n}{6} < 1$ that is $1 \leq k < \frac{6+2n}{3}$ then the minimum value is $c_4(1)$. Then we have

$$c_4(1) = \frac{2-3k-2n}{8-6k+8n-6kn}.$$

If $1 \leq \frac{3k-2n}{6} < k$ that is $k \geq \frac{6+2n}{3}$ then the minimum value is $c_4(\frac{3k-2n}{6})$ whenever $\frac{3k-2n}{6} \in \mathbf{Z}$.

$$c_4(\frac{3k-2n}{6}) = \frac{4}{-2+3k+2n}.$$

If $k \geq \frac{6+2n}{3}$ and $2 \nmid k$ and $3 \mid n$ then the minimum value is $c_4(\frac{3k-2n}{6} + \frac{1}{2}) = c_4(\frac{3k-2n}{6} - \frac{1}{2})$.

$$c_4(\frac{3k-2n}{6} + \frac{1}{2}) = \frac{4(-2+3k+2n)}{(-5+3k+2n)(1+3k+2n)}.$$

If $k \geq \frac{6+2n}{3}$ and $3 \nmid n$ and $2 \mid k$ then the minimum value is $c_4(\frac{3k-2n}{6} + \frac{1}{3}) = c_4(\frac{3k-2n}{6} - \frac{1}{3})$.

$$c_4(\frac{3k-2n}{6} - \frac{1}{3}) = \frac{4(-2+3k+2n)}{(-4+3k+2n)(3k+2n)}.$$

If $k \geq \frac{6+2n}{3}$ and $3 \nmid n$ and $2 \nmid k$ then the minimum value is $c_4(\frac{3k-2n}{6} + \frac{1}{6}) = c_4(\frac{3k-2n}{6} - \frac{1}{6})$.

$$c_4(\frac{3k-2n}{6} - \frac{1}{6}) = \frac{4(-2+3k+2n)}{(-3+3k+2n)(-1+3k+2n)}.$$

Case(v) Let $A_5 = \{x_i \mid 1 \leq i \leq n+1\}$ and $V \setminus A_5 = \{x_i \mid n+2 \leq i \leq n+k\} \cup \{y_i \mid 1 \leq i \leq n+k\}$. Then $vol(A_5) = 2n+2$ and $vol(V \setminus A_5) = 2n+6k-6$. Then we have $Ncut(A_5, V \setminus A_5) = 2\left(\dfrac{1}{2n+2} + \dfrac{1}{2n+6k-6}\right) = \dfrac{2n+3k-2}{(n+1)(n+3k-3)}$.
Now we can compare all cases considered above.
If $k = 2$ and $n = 1$ then it is easy show that c_1 is the minimum. If $k = 2$ and $n \geq 2$ then it is easy to show that c_2 is the minimum. If $k = 3$ and $n = 1$ then $c_4(\frac{3k-2n}{6} - \frac{1}{6})$ is the minimum. If $k = 3$ and $n = 2$ then $c_4(\frac{3k-2n}{6} + \frac{1}{6})$ is the minimum. If $k = 3$ and $n \geq 3$ then we can easily show that c_2 is the minimum. If $k \geq 4$ and $n = 1$ then c_4 is the minimum. Next we assume that $k \geq 4$ and $n \geq 2$. It is easy to check that c_2 is smaller than c_1, c_3 and c_5. So we compare c_2 with c_4 for $k \geq 4$. Then we have the following results. If ($*_1$ and $(n < K_1)$) then $c_4(\frac{3k-2n}{6})$ is smaller than c_2. If ($*_2$ and $(n < K_2)$) then $c_4(\frac{3k-2n}{6} - \frac{1}{6})$ is smaller than c_2. If ($*_3$ and $(n < K_3)$) then $c_4(\frac{3k-2n}{6} - \frac{1}{3})$ is smaller than c_2. If ($*_4$ and $(n < K_4)$) then $c_4(\frac{3k-2n}{6} + \frac{1}{2})$ is smaller than c_2. We can summarize the results as follows.

$$\begin{cases} c_1 & n=1, k=2, \\ c_4(\frac{3k-2n}{6}) & (*_1 \wedge (k \geq 4) \wedge (n < K_1)), \\ c_4(\frac{3k-2n}{6} + 1/2) & (*_4 \wedge (k \geq 4) \wedge (n < K_4)), \\ c_4(\frac{3k-2n}{6} - 1/3) & (*_3 \wedge (k \geq 4) \wedge (n < K_3)), \\ c_4(\frac{3k-2n}{6} - 1/6) & (*_2 \wedge (k \geq 4) \wedge (n < K_2)) \vee \\ & (n=1, k=3) \vee (n=2, k=3), \\ c_2 & ((k \geq 4) \wedge ((*_1 \wedge (K_1 \leq n)) \vee \\ & (*_2 \wedge (K_2 \leq n)))) \\ & \vee (*_3 \wedge (K_3 \leq n)) \vee (*_4 \wedge (K_4 \leq n)) \\ & \vee (k = 2 \wedge (n \geq 2)) \vee (k = 3 \wedge (n \geq 3)). \end{cases}$$

Finally, we want to show that for any arbitrary subset A, $cut(A, V \setminus A) = 1$ or $cut(A, V \setminus A) = 2$ gives the minimum normalized cut. We notice that every subset A with $cut(A, V \setminus A) = 1$ is A_2 or A_3 and every subset A with $cut(A, V \setminus A) = 2$ are A_1, A_5, A_4. We consider all cases with $cut(A, V \setminus A) = 1$ and the minimum occurs at A_2. There may be several partitions with $cut(A, V \setminus A) \geq 2$. Let $k \geq 4$. Then we note that $vol(R_{n,k}) \geq 24$ and there exists a subset A_4 in Case(iv), which minimize the $\left(\dfrac{1}{vol(A)} + \dfrac{1}{vol(R_{n,k}) - vol(A)}\right)$ with $cut(A, V \setminus A) = 2$. We note that $|vol(A_4) - \dfrac{vol(R_{n,k})}{2}| \leq 3$. From Lemma 11, $3\left(\dfrac{1}{vol(R_{n,k})/2} + \dfrac{1}{vol(R_{n,k})/2}\right) > 2\left(\dfrac{1}{vol(R_{n,k})/2+3} + \dfrac{1}{vol(R_{n,k})/2-3}\right)$ for $vol(R_{n,k}) \geq 11$. Then we can show that there is no subset A with $cut(A, V \setminus A) \geq 3$ and $Mcut(A, V \setminus A) \leq Mcut(A_4, V \setminus A_4)$. This conclude that minimum Ncut always have cut value 2 for all cases which has cut size more than 1. □

Figure 4 shows the above regions for n, k. For a given $R_{n,k}$, we can find $Mcut(R_{n,k})$.

3.4. *Mcut* of weighted paths $P_{n,k}$

In this section, we consider a weighted path graph $P_{n,k}$ and find a formula for $Mcut(P_{n,k})$ based on n, k. We consider subsets of

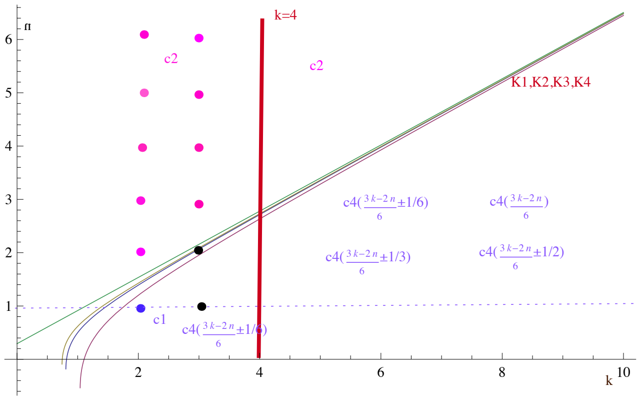


Figure 4: $Mcut(R_{n,k})$.

$V(P_{n,k})$ defined by $A(\alpha) = \{x_i \mid 1 \leq i \leq \alpha\}$ for $1 \leq \alpha \leq n+k-1$. We note that every subset $A \subset V(P_{n,k})$ with $cut(A, V \setminus A) = 1$ is $A = A(\alpha)$ for some α.

Lemma 13. *Let $G = P_{n,k}$. There exists a subset $A \subset V(P_{n,k})$ such that $cut(A, V \setminus A) = 1$ and $Mcut(G) = Ncut(A, V \setminus A)$.*

Proof. Since $vol(P_{n,k}) = 2n + 3k - 2$, if $k \geq \frac{1}{3}(11 - 2n)$ then $vol(P_{n,k}) \geq 9$. By the Lemma 10, we have $Mcut(G) = Mcut_1(G)$.

If $k < \frac{1}{3}(11 - 2n)$, we have only five cases $(n, k) = (1, 1)$, $(2, 1)$, $(3, 1)$, $(1, 2)$ and $(2, 2)$. For each cases $Mcut(P_{1,1}) = Ncut(A(1), V \setminus A(1))$, $Mcut(P_{2,1}) = Ncut(A(2), V \setminus A(2))$, $Mcut(P_{3,1}) = Ncut(A(2), V \setminus A(2))$, $Mcut(P_{1,2}) = Ncut(A(2), V \setminus A(2))$, and $Mcut(P_{2,2}) = Ncut(A(2), V \setminus A(2))$. □

Let $P_{n,k}$ $(k \geq \frac{1}{3}(11 - 2n))$ be a weighted path graph. We first note that

$$\begin{aligned} vol(P_{n,k}) &= 2n + 3k - 2, \\ vol(A(\alpha)) &= \begin{cases} 2\alpha - 1 & (\alpha \leq n) \\ 3\alpha - n - 1 & (n+1 \leq \alpha) \end{cases}, \text{and} \\ Ncut(A(\alpha), V \setminus A(\alpha)) &= c(\alpha), \end{aligned}$$

where a function $c(t)$ $(1 \leq t \leq n+k)$ is defined by

$$c(t) = \begin{cases} \frac{2n+3k-2}{(2t-1)(2n+3k-2t-1)} & (1 \leq t \leq n + \frac{1}{2}) \\ \frac{2n+3k-2}{(-n+3t-1)(3n+3k-3t-1)} & (n + \frac{1}{2} < t \leq n+k-1). \end{cases}$$

We note that $c(\alpha - x) = c(\alpha + x)$ for an integer α $(1 \leq \alpha \leq n$, $n+1 < \alpha \leq n+k-1)$ and a real number x $(0 \leq x \leq \frac{1}{2})$.

We also note $vol(A_i) < vol(A_{i+1})$ $(1 \leq i \leq n+k-2)$, $vol(A(n)) = 2n-1$, $vol(A(n+1)) = 2n+2$ and $vol(A(n+k-1)) = 2n+3k-4$. Since

$$\begin{aligned} & Ncut(A(\alpha), V \setminus A(\alpha)) \\ = & \frac{4vol(P_{n,k})}{(vol(P_{n,k}))^2 - (vol(A(\alpha)) - vol(V \setminus A(\alpha)))^2} \\ = & \frac{4vol(P_{n,k})}{(vol(P_{n,k}))^2 - 4(vol(A(\alpha)) - \frac{1}{2}vol(P_{n,k}))^2}, \end{aligned}$$

if $Mcut(P_{n,k}) = Ncut(A(\alpha_0), V \setminus A(\alpha_0))$ then

$$\begin{aligned} & |vol(A(\alpha_0)) - \frac{1}{2}vol(P_{n,k})| \\ = & \min\{|vol(A(\alpha)) - \frac{1}{2}vol(P_{n,k})| \mid 1 \leq \alpha \leq n+k-1\}. \end{aligned}$$

We consider four cases: Case (i) $\frac{1}{2}vol(P_{n,k}) \le vol(A(n))$, Case (ii) $vol(A(n)) < \frac{1}{2}vol(P_{n,k}) \le \frac{1}{2}(vol(A(n)) + vol(A(n+1)))$, Case (iii) $\frac{1}{2}(vol(A(n)) + vol(A(n+1))) < \frac{1}{2}vol(P_{n,k}) \le vol(A(n+1))$, and Case (iv) $vol(A(n+1)) < \frac{1}{2}vol(P_{n,k}) < vol(A(n))$.

Case (i) Assume $\frac{1}{2}vol(P_{n,k}) \le vol(A(n))$. That is $k \le \frac{2}{3}n$. We find α minimizing $|vol(A(\alpha)) - \frac{1}{2}vol(P_{n,k})| = |2\alpha - 1 - (n + \frac{3}{2}k - 1)|$. For such α we have

$$(2\alpha - 1) - 1 < n + \frac{3}{2}k - 1 \le (2\alpha - 1) + 1.$$

That is

$$\alpha - \frac{1}{2} < \frac{2n+3k}{4} \le \alpha + \frac{1}{2}$$

which means α is the nearest integer of $\frac{2n+3k}{4}$.

We consider three cases ($K \in \mathbf{Z}$), ($K \notin \mathbf{Z}$ and $2 \mid k$), and ($2 \nmid k$), where $K = \frac{2n+3k}{4}$.

If $K \in \mathbf{Z}$ then $\alpha = K$. If $2 \nmid k$ then $\alpha = K + \frac{1}{4}$ or $\alpha = K - \frac{1}{4}$. If $K \notin \mathbf{Z}$ and $2 \mid k$ then $\alpha = K + \frac{1}{2}$ or $\alpha = K - \frac{1}{2}$.

Since $c(\alpha - x) = c(\alpha + x)$ for an integer α ($1 \le \alpha \le n$) and a real number x ($0 \le x \le \frac{1}{2}$), $Mcut(P_{n,k})$ will be

$$\begin{aligned} c(K) &= \frac{4}{-2+3k+2n}, \\ c(K+\frac{1}{2}) &= \frac{4(-2+3k+2n)}{(-4+3k+2n)(3k+2n)}, or \\ c(K+\frac{1}{4}) &= \frac{4(-2+3k+2n)}{(-3+3k+2n)(-1+3k+2n)}. \end{aligned}$$

following the conditions of n and k.

Case (ii) Assume $vol(A(n)) < \frac{1}{2}vol(P_{n,k}) \le \frac{1}{2}(vol(A(n)) + vol(A(n+1)))$. That is $\frac{2}{3}n < k \le \frac{2}{3}n + 1$. In this case

$$Mcut(P_{n,k}) = c(n) = \frac{3k+2n-2}{(3k-1)(2n-1)}.$$

Case (iii) Assume $\frac{1}{2}(vol(A(n)) + vol(A(n+1))) < \frac{1}{2}vol(P_{n,k}) \le vol(A(n+1))$. That is $\frac{2}{3}n + 1 < k \le \frac{2}{3}n + 2$. In this case

$$Mcut(P_{n,k}) = c(n+1) = \frac{2-3k-2n}{8-6k+8n-6kn}.$$

Case (iv) Assume $vol(A(n+1)) < \frac{1}{2}vol(P_{n,k}) < vol(A(n))$. That is $\frac{2}{3}n + 2 < k$. We find α minimizing $|vol(A(\alpha)) - \frac{1}{2}vol(P_{n,k})| = |3\alpha - n - 1 - (n + \frac{3}{2}k - 1)|$. For such α we have

$$(3\alpha - n - 1) - \frac{3}{2} < n + \frac{3}{2}k - 1 \le (3\alpha - n - 1) + \frac{3}{2}.$$

That is

$$\alpha - \frac{1}{2} < \frac{4n+3k}{6} \le \alpha + \frac{1}{2}$$

which means α is the nearest integer of $\frac{4n+3k}{6}$.

We consider four cases ($K' \in \mathbf{Z}$), ($3 \nmid n$ and $2 \mid k$), ($3 \mid n$ and $2 \nmid k$), and ($3 \nmid n$ and $2 \nmid k$), where $K' = \frac{4n+3k}{6}$.

If $K' \in \mathbf{Z}$ then $\alpha = K'$. If $3 \nmid n$ and $2 \mid k$ then $\alpha = K' + \frac{1}{3}$ or $\alpha = K' - \frac{1}{3}$. If $3 \mid n$ and $2 \nmid k$ then $\alpha = K' + \frac{1}{2}$ or $\alpha = K' - \frac{1}{2}$. If $3 \nmid n$ and $2 \nmid k$ then $\alpha = K' + \frac{1}{6}$ or $\alpha = K' - \frac{1}{6}$. Since $c(K' - x) = c(K' + x)$ ($x = \frac{1}{2}, \frac{1}{3}, \frac{1}{6}$), we have $Mcut(P_{n,k})$ as one of

$$\begin{aligned} c(K') &= \frac{4}{-2+3k+2n}, \\ c(K'+\frac{1}{2}) &= \frac{4(-2+3k+2n)}{(-5+3k+2n)(1+3k+2n)}, \\ c(K'+\frac{1}{3}) &= \frac{4(-2+3k+2n)}{(-4+3k+2n)(3k+2n)}, or \\ c(K'+\frac{1}{6}) &= \frac{4(-2+3k+2n)}{(-3+3k+2n)(-1+3k+2n)} \end{aligned}$$

following the conditions of n and k.

We note $c(K) = c(K')$, $c(K + \frac{1}{2}) = c(K' + \frac{1}{3})$ and $c(K + \frac{1}{4}) = c(K' + \frac{1}{6})$ before summarizing them as a proposition.

Theorem 3. *For $P_{n,k}$,$(n, k \ge 1,\ k \ge \frac{1}{3}(11 - 2n))$, $Mcut(P_{n,k})$ is given by*

$$\begin{cases} \frac{4}{-2+3k+2n} & (((3 \mid n) \wedge (2 \mid k) \wedge (R_3 < k)) \vee \\ & \quad (o_1 \wedge (k \le R_1)) \\ \frac{4(-2+3k+2n)}{(-5+3k+2n)(1+3k+2n)} & ((3 \mid n) \wedge (2 \nmid k) \wedge (R_3 < k)), \\ \frac{4(-2+3k+2n)}{(-4+3k+2n)(3k+2n)} & ((3 \nmid n) \wedge (2 \mid k) \wedge (R_3 < k)) \vee \\ & \quad (o_2 \wedge (2 \mid k) \wedge (k \le R_1)), \\ \frac{4(-2+3k+2n)}{(-3+3k+2n)(-1+3k+2n)} & ((3 \nmid n) \wedge (2 \nmid k) \wedge (R_3 < k)) \vee \\ & \quad ((2 \nmid k) \wedge (k \le R_1)) \\ \frac{3k+2n-2}{(3k-1)(2n-1)} & (R_1 < k \le R_2) \\ \frac{2-3k-2n}{8-6k+8n-6kn} & (R_2 < k \le R_3), \end{cases}$$

where

$$\begin{aligned} o_1 &= (\frac{3k+2n}{4} \in \mathbf{Z}), \\ o_2 &= (\frac{3k+2n}{4} \notin \mathbf{Z}), \\ R_1 &= \frac{2n}{3}, \\ R_2 &= \frac{2n}{3} + 1, \\ R_3 &= \frac{2n}{3} + 2. \end{aligned}$$

□

Figure 5 shows minimum $Mcut(P_{n,k})$ for each n, k.

Corollary 1. *For $P_{2k,k}$,*

$$Mcut(P_{2k,k}) = \begin{cases} \frac{4}{-2+7k} & (4 \mid k), \\ \frac{4(-2+7k)}{(-4+7k)(7k)} & (4 \nmid k) \wedge (2 \mid k), \\ \frac{4(-2+7k)}{(-3+7k)(-1+7k)} & (2 \nmid k). \end{cases}$$

Proof. By substituting $n = 2k$ to the formula given for $Mcut(P_{n,k})$, we can directly obtain the result. According to the

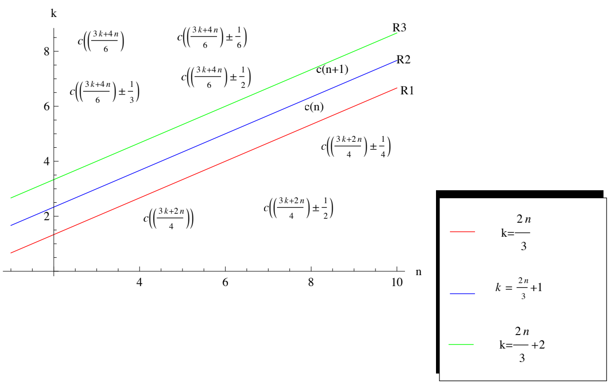


Figure 5: $Mcut(P_{n,k})$.

Theorem 3, for $n = 2k$, $k > R_3$ that is $k > \frac{2n}{3} + 2$ implies that $k \leq -6$. Since $k \geq 1$, this does not holds. For $R_1 < k \leq R_2$ that is $\frac{2n}{3} < k \leq \frac{2n}{3} + 1$ implies that $-3 \leq k < 0$. Since $k \geq 1$, this does not holds. For $R_2 < k \leq R_3$ that is $\frac{2n}{3} + 1 < k \leq \frac{2n}{3} + 2$ implies that $-6 \leq k < -3$. Since $k \geq 1$, this does not holds. Therefore the only case, which holds for $n = 2k$ is, $k \leq R_1$ that is $k \leq \frac{4k}{3}$. This implies that $k \geq 0$. Substituting $n = 2k$ in the Theorem 3, we have,

$$Mcut(P_{2k,k}) = \begin{cases} \frac{4}{-2+7k} & (4 \mid k), \\ \frac{4(-2+7k)}{(-4+7k)(7k)} & (4 \nmid k) \wedge (2 \mid k), \\ \frac{4(-2+7k)}{(-3+7k)(-1+7k)} & (2 \nmid k). \end{cases}$$

□

3.5. *Mcut* of graph $LP_{n,m}$

Here, we consider lollipop graph $LP_{n,m}$ and derive a formula for $Mcut(LP_{n,m})$. A lollipop graph $LP_{n,m}$ defined in Definition 33 is constructed by joining an end vertex of a path graph P_m to a vertex of a complete graph K_n.

We consider three kinds of subsets of $V(LP_{n,m})$ defined by $A_1(\alpha) = \{x_i \mid 1 \leq i \leq \alpha\}$ for $1 \leq \alpha \leq m$, $A_2(\beta) = \{x_i \mid 1 \leq i \leq m\} \cup \{y_i \mid 1 \leq i \leq \beta\}$ for $1 \leq \beta < n$, and, $B(\alpha, \beta) = \{x_i \mid 1 \leq i \leq \alpha\} \cup \{x_m\} \cup \{y_i \mid 1 \leq i \leq \beta\}$ for $1 \leq \alpha < m - 1$, $1 \leq \beta < n$.

Lemma 14. *Let A be a subset of $V(LP_{n,m})$.*

1. *If $y_i \in A$ and $y_{i+1} \notin A$ for some i ($2 \leq i \leq n-1$) then $Ncut(A', V \setminus A') = Ncut(A, V \setminus A)$, where $A' = (A \setminus \{y_i\}) \cup \{y_{i+1}\}$.*
2. *If $x_i \in A$, $x_{i+1}, \ldots, x_j \notin A$, and $x_{j+1} \in A$ for some i, j ($1 \leq i < j \leq m-1$) then $Ncut(A', V \setminus A') \leq Ncut(A, V \setminus A)$, where $A' = (A \setminus \{x_{j+1}\}) \cup \{x_{i+1}\}$.*
3. *There exists a subset $A_1(\alpha)$, $A_2(\beta)$ or $B(\alpha, \beta)$ such that $Mcut(LP_{n,m}) = Ncut(A_1(\alpha), V \setminus A_1(\alpha))$, $Mcut(LP_{n,m}) = Ncut(A_2(\beta), V \setminus A_2(\beta))$, or $Mcut(LP_{n,m}) = Ncut(B(\alpha, \beta), V \setminus B(\alpha, \beta))$.*

Proof.
1. It is easy to check $vol(A) = vol(A')$ and $cut(A, V \setminus A) = cut(A', V \setminus A')$.
2. It is easy to check $vol(A) = vol(A')$ and $cut(A', V \setminus A') \leq cut(A, V \setminus A)$.
3. Let A be a subset of $V(LP_{n,m})$ such that $Mcut(LP_{n,m}) = Ncut(A, V \setminus A)$. Using the above results 1. and 2., we have a subset A' which is one of $A_1(\alpha)$, $A_2(\alpha)$ or $B(\alpha, \beta)$ such that $Ncut(A', V \setminus A') = Mcut(LP_{n,m})$.

□

Let $G = LP_{n,m}$ ($n \geq 3$, $m \geq 1$) a lollipop graph. We first note that

$$\begin{aligned} vol(LP_{n,m}) &= 2m + n(n-1), \\ vol(A_1(\alpha)) &= 2\alpha - 1, \\ cut(A_1(\alpha), V \setminus A_1(\alpha)) &= 1, \\ vol(A_2(\beta)) &= 2m + \beta(n-1), \\ cut(A_2(\beta), V \setminus A_2(\beta)) &= \beta(n - \beta), \\ vol(B(\alpha, \beta)) &= 2\alpha + 2 + \beta(n-1), \\ cut(B(\alpha, \beta, V \setminus B(\alpha, \beta))) &= \beta(n - \beta) + 2, \textit{and} \\ Ncut(A_1(\alpha), V \setminus A_1(\alpha)) &= c(\alpha), \end{aligned}$$

where a function $c(t)$ ($1 \leq t \leq m$) is defined by

$$c(t) = \frac{2m - n + n^2}{(1 + 2m - n + n^2 - 2t)(-1 + 2t)}.$$

It is also showed that

$$\begin{aligned} & Ncut(A_2(\beta), V \setminus A_2(\beta)) \\ = \; & \frac{\beta\left(2m - n + n^2\right)}{(-1+n)(-\beta + 2m + n\beta)}, \textit{and} \\ & Ncut(B(\alpha, \beta), V \setminus B(\alpha, \beta)) \\ = \; & \frac{\left(2m - n + n^2\right)\left(2 + n\beta - \beta^2\right)}{(2 + 2\alpha - \beta + n\beta)(2m - n + n^2 - 2\alpha + \beta - n\beta - 2)}. \end{aligned}$$

Lemma 15. *Let $G = LP_{n,m}$ ($n \geq 3$, $m \geq 2$).*

1. *$c(\alpha - 1)) < c(\alpha)$ iff $m > \frac{1}{2}(n^2 - n + 4)$ ($2 \leq \alpha \leq m$).*
2. *$c(m) \leq \frac{1}{2} vol(LP_{n,m})$ iff $m \leq \frac{1}{2}(n^2 - n + 4)$.*
3. *$c(m) \leq Ncut(A_2(\beta), V \setminus A_2(\beta))$ ($1 \leq \beta < n$).*
4. *If $m \leq \frac{1}{2}(n^2 - n + 2)$ then*
$$c(m) \leq Ncut(B(\alpha, \beta), V \setminus B(\alpha, \beta)),$$
($1 \leq \alpha \leq m - 2$, $1 \leq \beta < n$).

Proof. Each items are given by straightforward computations. □

Since $cut(A_1(\alpha), V \setminus A_1(\alpha)) = 1$, if $vol(A_1(m)) \geq \frac{1}{2} vol(LP_{n,m})$ then there exists some α such that $Mcut(LP_{n,m}) = Ncut(A_1(\alpha), V \setminus A_1(\alpha))$. To find the α, we solve

$$vol(A_1(\alpha)) - 1 < \frac{1}{2} vol(LP_{n,m}) \leq vol(A_1(\alpha)) + 1.$$

That is

$$\alpha - \frac{1}{2} < \frac{n^2 - n + 2m + 2}{4} \leq \alpha + \frac{1}{2}$$

which means α is the nearest integer of $\frac{n^2-n+2m+2}{4}$. We consider two cases ($K \in \mathbf{Z}$) and ($K \notin \mathbf{Z}$), where $K = \frac{n^2-n+2m+2}{4}$. If $K \in \mathbf{Z}$ then $\alpha = K$. If $K \notin \mathbf{Z}$ then $K + \frac{1}{2}$ is an integer and $\alpha = K + \frac{1}{2}$ or $\alpha = K - \frac{1}{2}$. Since $c(K + \frac{1}{2}) = c(K - \frac{1}{2})$, $Mcut(LP_{n,m})$ will be

$$\begin{aligned} c(K) &= \frac{4}{n^2 - n + 2m}, \textit{ or} \\ c(K + \frac{1}{2}) &= \frac{4(n^2 - n + 2m)}{(n(n-1) + 2(m-1))(n(n-1) + 2(m+1))} \end{aligned}$$

By Lemma 15, if $m \leq \frac{1}{2}(n^2 - n + 4)$ then $Mcut(LP_{n,m}) = Ncut(A_1(m), V \setminus A_1(m))$. That is

$$Ncut(A_1(m), V \setminus A_1(m)) = \frac{n^2 - n + 2m}{(2m-1)(n^2 - n + 1)}.$$

If $m = 1$ then it is easy to verify $Mcut(LP_{n,1}) = Ncut(A_2(1), V \setminus A_2(1)) = \frac{n^2-n+2}{(n+1)(n-1)}$.

Theorem 4. *For the graph $LP_{n,m}, (n \geq 3$ and $m \geq 1)$, $Mcut(LP_{n,m})$ is given by,*

$$\begin{cases} \frac{n^2-n+2m}{(2m-1)(n^2-n+1)} & (2 \leq m \leq \frac{n^2-n+4}{2}), \\ \frac{4}{(n^2-n+2m)} & (o_1 \wedge m > \frac{n^2-n+4}{2}), \\ \frac{4(n^2-n+2m)}{(n(n-1)+2(m-1))(n(n-1)+2(m+1))} & (o_2 \wedge m > \frac{n^2-n+4}{2}), \\ \frac{n^2-n+2}{(n+1)(n-1)} & (m = 1), \end{cases}$$

where

$$\begin{aligned} o_1 &= (\frac{n^2 - n + 2m + 2}{4} \in \mathbf{Z}), \\ o_2 &= (\frac{n^2 - n + 2m + 2}{4} \notin \mathbf{Z}). \end{aligned}$$

□

4. Eigenvalues and eigenvectors of paths and cycles

In this section, we derive formulae for the eigenvalues and eigenvectors of cycles and paths using circulant matrices and give an alternate proof for the eigenvalues of adjacency matrix of cycles and paths using Chebyshev polynomials.

4.1. Circulant matrices and eigenvalues of cycles and paths

Let $\omega_n = \mathbf{e}^{-\frac{2\pi}{n}\mathbf{i}} = \cos\frac{2\pi}{n} + \mathbf{i}\sin\frac{2\pi}{n}$ be a primitive n-th root of unity.

Definition 36. A circulant matrix $C = (c_{ij})$ is a matrix having a form $c_{ij} = c_{(j-i) \mod n}$.

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & c_2 & & c_{n-2} \\ \vdots & c_{n-1} & c_0 & c_1 & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & c_1 \\ c_1 & c_2 & \cdots & \cdots & c_{n-1} & c_0 \end{pmatrix}.$$

Proposition 10. *Let $C = (c_{ij})$ be a circulant matrix and $c_{ij} = c_{(j-i) \mod n}$. For $k = 0, \ldots, n-1$, we have*

$$C\mathbf{u}_k = \lambda_k \mathbf{u}_k,$$

where $\lambda_k = \sum_{j=0}^{n-1} c_j(\omega_n^k)^j$, $\mathbf{u}_k = (u_{ki})$ and $u_{ki} = (\omega_n^k)^i = \cos\frac{2k\pi i}{n} + \mathbf{i}\sin\frac{2k\pi i}{n}$.

Proof.

$$\begin{aligned} (C\mathbf{u}_k)_i &= \sum_{j=0}^{n-1} c_{ij}u_{kj} \\ &= \sum_{j=0}^{n-1} c_{(j-i) \mod n}(\omega_n^k)^j \\ &= (\omega_n^k)^i \sum_{j=0}^{n-1} c_{(j-i) \mod n}(\omega_n^k)^{j-i} \\ &= (\omega_n^k)^i \sum_{j=0}^{n-1} c_j(\omega_n^k)^j \\ &= \lambda_k u_{ki} \\ &= (\lambda_k \mathbf{u}_k)_i. \end{aligned}$$

□

Proposition 11. *1. The eigenvalues of the adjacency matrix of C_n is given by $\lambda_k = 2\cos(\frac{2k\pi}{n})$,*

2. The eigenvalues of the difference Laplacian matrix of C_n is given by $\lambda_k = 2 - 2\cos(\frac{2k\pi}{n})$,

3. The eigenvalues of the normalized Laplacian matrix of C_n is given by $\lambda_k = 1 - \cos(\frac{2k\pi}{n})$, and

4. The eigenvalues of the signless Laplacian matrix of C_n is given by $\lambda_k = 2 + 2\cos(\frac{2k\pi}{n})$, where $k = (0, \ldots, n-1)$.

Proof. 1. Let A be an adjacency matrix of a cycle graph with n vertices. That is $A = (c_{ij}) = c_{(j-i) \mod n}$ and $c_0 = 0$, $c_1 = c_{n-1} = 1$ and $c_i = 0$ for $i = 2, \ldots, n-2$.

$$\begin{aligned} \lambda_k &= (\omega_n^k)^1 + (\omega_n^k)^{n-1} \\ &= (\omega_n^k)^1 + (\omega_n^k)^{-1} \\ &= 2\cos(\frac{2k\pi}{n}). \end{aligned}$$

□

2. Let $L(C_n)$ be the Laplacian matrix of a cycle graph with n vertices. That is $L(C_n) = (c_{ij}) = c_{(j-i) \mod n}$ and $c_0 = 2$, $c_1 = c_{n-1} = -1$ and $c_i = 0$ for $i = 2, \ldots, n-2$.

$$\begin{aligned} \lambda_k &= 2 - (\omega_n^k)^1 - (\omega_n^k)^{n-1} \\ &= 2 - ((\omega_n^k)^1 + (\omega_n^k)^{-1}) \\ &= 2 - 2\cos(\frac{2k\pi}{n}). \end{aligned}$$

□

3. Let $\mathcal{L}(C_n)$ be the normalized Laplacian matrix of a cycle graph with n vertices. That is $\mathcal{L}(C_n) = (c_{ij}) = c_{(j-i) \bmod n}$ and $c_0 = 1, c_1 = c_{n-1} = -\frac{1}{2}$ and $c_i = 0$ for $i = 2, \ldots, n-2$.

$$\begin{aligned}\lambda_k &= 1 - \frac{1}{2}(\omega_n^k)^1 - \frac{1}{2}(\omega_n^k)^{n-1} \\ &= 1 - \frac{1}{2}((\omega_n^k)^1 + (\omega_n^k)^{-1}) \\ &= 1 - \cos(\frac{2k\pi}{n}).\end{aligned}$$

□

4. Let $SL(C_n)$ be the signless Laplacian matrix of a cycle graph with n vertices. That is $SL(C_n) = (c_{ij}) = c_{(j-i) \bmod n}$ and $c_0 = 2, c_1 = c_{n-1} = 1$ and $c_i = 0$ for $i = 2, \ldots, n-2$.

$$\begin{aligned}\lambda_k &= 2 + (\omega_n^k)^1 + (\omega_n^k)^{n-1} \\ &= 2 + ((\omega_n^k)^1 + (\omega_n^k)^{-1}) \\ &= 2 + 2\cos(\frac{2k\pi}{n}). \quad \square\end{aligned}$$

□

Proposition 12. *Let $\lambda_k(0 \le k \le n-1)$ be the k^{th} eigenvalue of an adjacency matrix of C_n. Then $\lambda_k = \lambda_{n-k}$, for $k = 1, \ldots, n-1$.*

Proof. Eigenvalues of an adjacency matrix of cycle is given by $\lambda_k = 2\cos(\frac{2k\pi}{n})$, where $k = 0, \ldots, n-1$.

$$\begin{aligned}\lambda_0 &= 2, \\ \lambda_1 &= 2\cos(\frac{2\pi}{n}), \\ \lambda_2 &= 2\cos(\frac{4\pi}{n}), \\ &\vdots \\ \lambda_{n-2} &= 2\cos(\frac{4\pi}{n}), \\ \lambda_{n-1} &= 2\cos(\frac{2\pi}{n}).\end{aligned}$$

This shows that $\lambda_k = \lambda_{n-k}$ for $k = 1, \ldots, n-1$. □

Proposition 13. *1. The eigenvalues of an adjacency matrix of a path graph P_n are given by $\lambda_k(A(P_n)) = 2\cos(\frac{(k+1)\pi}{n+1}), (k = 0, \ldots, n-1)$ and an eigenvector $\mathbf{u_k}$ is given by $(u_{ki}) = \sin\frac{(i+1)(k+1)\pi}{n+1}, (i = 0, \ldots, n-1)$ and $(k = 0, \ldots, n-1)$.*

2. The eigenvalues of difference Laplacian matrix of P_n are given by $\lambda_k(L(P_n)) = 2 - 2\cos(\frac{k\pi}{n})$, $(k = 0, \ldots, n-1)$ and its eigenvector $\mathbf{u_k}$ is given by $(u_{ki}) = \cos\left(\frac{(2i+1)k\pi}{2n}\right)(i = 0, \ldots, n-1)$.

3. The eigenvalues of normalized Laplacian matrix of a path P_n are given by $\lambda_k(\mathcal{L}(P_n)) = 1 - \cos(\frac{k\pi}{n-1})$ $(k = 0, \ldots, n-1)$ and its eigenvector $\mathbf{u_k}$ is given by

$$u_{ki} = \begin{cases} \sqrt{2}\cos\left(\frac{2i\pi k}{2n-2}\right) & i = 1, \ldots, n-2, \\ \cos(\frac{2i\pi k}{2n-2}) & i = 0 \text{ and } i = n-1. \end{cases}$$

4. The eigenvalues of signless Laplacian matrix of P_n are given by $\lambda_k(SL(P_n)) = 2 + 2\cos(\frac{(k+1)\pi}{n})$, $(k = 0, \ldots, n-1)$ and its eigenvector $\mathbf{u_k}$ is given by $(u_{ki}) = \sin(\frac{(2i+1)k\pi}{2n}), (i = 0, \ldots, n-1)$.

Proof. 1. Let $\mathbf{u} = (u_i), (i = 0, \ldots, n-1)$ be an eigenvector for an eigenvalue λ of path P_n. Then, we can write

$$P_n\mathbf{u} = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 1 & 0 & & \vdots \\ 0 & 1 & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_{n-1} \end{pmatrix} = \lambda\mathbf{u}.$$

Then we have the following equations:

$$\begin{aligned}u_1 &= \lambda u_0, \\ u_0 + u_2 &= \lambda u_1, \\ u_1 + u_3 &= \lambda u_2, \\ &\vdots \\ u_{n-2} &= \lambda u_{n-1}.\end{aligned} \tag{1}$$

Let $\mathbf{u'} = (u'_i), (i = 0, \ldots, 2n+1)$ be an eigenvector of C_{2n+2}, where $(u'_i) = \sin\left(\frac{2(i+1)(k+1)\pi}{2n+2}\right), (i = 0, \ldots, 2n+1)$ and $(k = 0, \ldots, 2n+1)$. The eigenvalues of an adjacency matrix of a cycle C_{2n+2} are $\lambda_k = 2\cos\left(\frac{(k+1)\pi}{n+1}\right), (k = 0, \ldots, 2n+1)$. We note that $u'_n = u'_{2n+1} = 0$. Hence we can write the equation $C_{2n+2}\mathbf{u'} = \lambda_k\mathbf{u'}$ as

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 1 \\ 1 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & & & \vdots \\ \vdots & & 1 & 0 & 1 & & & \vdots \\ \vdots & & & 1 & 0 & 1 & & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & & 1 & 0 & 1 \\ 1 & 0 & \cdots & \cdots & \cdots & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} u'_0 \\ \vdots \\ u'_{n-1} \\ 0 \\ u'_{n+1} \\ \vdots \\ u'_{2n} \\ 0 \end{pmatrix}$$

$= \lambda_k\mathbf{u'}$.

Then we have the following equations:

$$\begin{aligned}u'_1 &= \lambda_k u'_0, \\ u'_0 + u'_2 &= \lambda_k u'_1, \\ &\vdots \\ u'_{n-2} &= \lambda_k u'_{n-1}.\end{aligned} \tag{2}$$

Comparing Equation 1 with Equation 2, we have $P_n\mathbf{u} = \lambda_k\mathbf{u}$, where $\mathbf{u} = (u'_i)(i = 0,\ldots,n-1)$. That is $\lambda_k, (k = 0,\ldots,n-1)$ are eigenvalues of P_n and $\mathbf{u}$ is an eigenvector of λ_k. Since $\lambda_i \neq \lambda_j$ for ($i \neq j$ and $0 \leq i, j \leq n-1$), we have n different eigenvalues of P_n and that is the complete set of eigenvalues of P_n.

2. Let $\mathbf{u} = (u_i), (i = 0,\ldots,n-1)$ be an eigenvector for an eigenvalue λ of difference Laplacian matrix $L(P_n)$. Then we can write the equation $L(P_n)\mathbf{u} = \lambda\mathbf{u}$ as

$$\begin{pmatrix} 1 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & & 0 \\ 0 & -1 & 2 & -1 & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & -1 & 2 & -1 \\ 0 & 0 & \cdots & \cdots & -1 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{pmatrix} = \lambda\mathbf{u}.$$

Then we have the following equations.

$$\begin{aligned} u_0 - u_1 &= \lambda u_0, \\ -u_0 + 2u_1 - u_2 &= \lambda u_1, \\ &\vdots \\ -u_{n-2} + u_{n-1} &= \lambda u_{n-1}. \end{aligned} \tag{3}$$

Let $\mathbf{u}' = (u'_i), (i = 0,\ldots,2n-1)$ be an eigenvector of difference Laplacian matrix of C_{2n}, where $(u'_i) = \cos\left(\dfrac{(2i+1)k\pi}{2n}\right), (i = 0,\ldots,2n-1)$ and $(k = 0,\ldots,2n-1)$. The eigenvalues of $L(C_{2n})$ are $\lambda_k = 2 - 2\cos(\dfrac{k\pi}{n}), (k = 0,\ldots,2n-1)$. We note that $u'_0 = u'_{2n-1}, u'_1 = u'_{2n-2},\ldots,u'_{n-1} = u'_n$.
Then we can write the equation $L(C_{2n})\mathbf{u}' = \lambda_k\mathbf{u}'$ as

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & -1 \\ -1 & 2 & -1 & 0 & & 0 \\ 0 & -1 & 2 & -1 & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & -1 & 2 & -1 \\ -1 & 0 & \cdots & \cdots & -1 & 2 \end{pmatrix} \begin{pmatrix} u'_0 \\ u'_1 \\ \vdots \\ \vdots \\ u'_{2n-2} \\ u'_{2n-1} \end{pmatrix} = \lambda_k\mathbf{u}'.$$

$$\begin{aligned} 2u'_0 - u'_1 - u'_{2n-1} &= u'_0 - u'_1 = \lambda_k u'_0, \\ -u'_0 + 2u'_1 - u'_2 &= \lambda_k u'_1, \\ &\vdots \\ -u'_{n-2} + 2u'_{n-1} - u'_n &= -u'_{n-2} + u'_{n-1} = \lambda_k u'_{n-1}. \end{aligned} \tag{4}$$

Comparing Equation 3 and Equation 4, we have $P_n\mathbf{u} = \lambda_k\mathbf{u}$, where $\mathbf{u} = (u'_i), (i = 0,\ldots,n-1)$. That is $\lambda_k, (k = 0,\ldots,n-1)$ are eigenvalues of P_n and $\mathbf{u}$ is an eigenvector of λ_k. Since $\lambda_i \neq \lambda_j$ for ($i \neq j$ and $0 \leq i, j \leq n-1$), we have n different eigenvalues of P_n and that is the complete set of eigenvalues of P_n.

3. Let $\mathbf{u} = (u_i), (i = 0,\ldots,n-1)$ be an eigenvector for an eigenvalue λ of normalized Laplacian matrix of path P_n. Then we can write the equation $\mathcal{L}(P_n)\mathbf{u} = \lambda\mathbf{u}$ as

$$\begin{pmatrix} 1 & -\frac{1}{\sqrt{2}} & 0 & \cdots & \cdots & 0 \\ -\frac{1}{\sqrt{2}} & 1 & -\frac{1}{2} & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{2}} \\ \vdots & \cdots & \cdots & \cdots & -\frac{1}{\sqrt{2}} & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{pmatrix}$$

$= \lambda\mathbf{u}$.

By expanding this we have the following equations.

$$\begin{aligned} u_0 - \frac{1}{\sqrt{2}}u_1 &= \lambda u_0, \\ -\frac{1}{\sqrt{2}}u_0 + u_1 - \frac{1}{2}u_2 &= \lambda u_1, \\ &\vdots \\ -\frac{1}{2}u_{n-3} + u_{n-2} - \frac{1}{\sqrt{2}}u_{n-1} &= \lambda u_{n-2}, \\ -\frac{1}{\sqrt{2}}u_{n-2} + u_{n-1} &= \lambda u_{n-1}. \end{aligned} \tag{5}$$

Let $\mathbf{u}' = (u'_i), (i = 0,\ldots,2n-3)$ be an eigenvector of normalized Laplacian matrix of C_{2n-2}, where $(u'_i) = \cos(\frac{2ik\pi}{2n-2}), (i = 0,\ldots,2n-3)$ and $\lambda_k = 1 - \cos(\frac{2k\pi}{2n-2}), (k = 0,\ldots,2n-3)$ be its eigenvalue. We note that $u'_1 = u'_{2n-3}, u'_2 = u'_{2n-4},\ldots,u'_{n-2} = u'_n$. Then we multiply each of these values by $\frac{1}{\sqrt{2}}$ and obtain the vector, $u'_0, \frac{1}{\sqrt{2}}u'_1, \frac{1}{\sqrt{2}}u'_2,\ldots,\frac{1}{\sqrt{2}}u'_{n-2}, u'_{n-1}, \frac{1}{\sqrt{2}}u'_n,\ldots,\frac{1}{\sqrt{2}}u'_{2n-3}$. We can write $\mathcal{L}(C_{2n-2})\mathbf{u}' = \lambda_k\mathbf{u}'$ as

$$\begin{pmatrix} 1 & -\frac{1}{2} & \cdots & \cdots & \cdots & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & \cdots & \cdots & \cdots & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} u'_0 \\ \frac{1}{\sqrt{2}}u'_1 \\ \vdots \\ u'_{n-1} \\ \vdots \\ \frac{1}{\sqrt{2}}u'_{2n-3} \end{pmatrix}$$

$= \lambda_k\mathbf{u}'$.

By expanding we have,

$$\begin{aligned}
u'_0 - \frac{1}{2}\frac{1}{\sqrt{2}}u'_1 - \frac{1}{2}\frac{1}{\sqrt{2}}u'_1 &= u'_0 - \frac{1}{\sqrt{2}}u'_1 \\
&= \lambda_k u'_0, \\
-\frac{1}{2}u'_0 + \frac{1}{\sqrt{2}}u'_1 - \frac{1}{2}\frac{1}{\sqrt{2}}u'_2 &= \frac{1}{\sqrt{2}}(-\frac{1}{\sqrt{2}}u'_0 + u'_1 - \frac{1}{2}u'_2) \\
&= \lambda_k(\frac{1}{\sqrt{2}}u'_1), \qquad (6) \\
&\vdots \\
-\frac{1}{2}\frac{1}{\sqrt{2}}u'_{n-3} + \frac{1}{\sqrt{2}}u'_{n-2} - \frac{1}{2}u'_{n-1} &= \lambda_k(\frac{1}{\sqrt{2}}u'_{n-2}), \\
-\frac{1}{2}\frac{1}{\sqrt{2}}u'_{n-2} + u'_{n-1} - \frac{1}{2}\frac{1}{\sqrt{2}}u'_{n-2} &= -\frac{1}{\sqrt{2}}u'_{n-2} + u'_{n-1} \\
&= \lambda_k u'_{n-1}.
\end{aligned}$$

Comparing Equation 5 and Equation 6, we have $P_n\mathbf{u} = \lambda_k\mathbf{u}$, where $\mathbf{u} = (u'_0, \sqrt{2}u'_1, \dots, \sqrt{2}u'_{n-2}, u'_{n-1})$. That is $\lambda_k, (k = 0, \dots, n-1)$ are eigenvalues of P_n and $\mathbf{u}$ is an eigenvector of λ_k. Since $\lambda_i \neq \lambda_j$ for ($i \neq j$ and $0 \leq i, j \leq n-1$), we have n different eigenvalues of P_n and that is the complete set of eigenvalues of P_n.

4. Let $\mathbf{u} = (u_i), (i = 0, \dots, n-1)$ be an eigenvector for an eigenvalue λ of signless Laplacian matrix of path P_n. Then we can write the equation $SL(P_n)\mathbf{u} = \lambda\mathbf{u}$ as

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 2 & 1 & 0 & & \vdots \\ 0 & 1 & 2 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 1 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{pmatrix} = \lambda\mathbf{u}$$

.

$$\begin{aligned}
u_0 + u_1 &= \lambda u_0, \\
u_0 + 2u_1 + u_2 &= \lambda u_1, \\
&\vdots \qquad (7) \\
u_{n-2} + u_{n-1} &= \lambda u_{n-1}.
\end{aligned}$$

Let $\mathbf{u}' = (u'_i), (i = 0, \dots, 2n-1)$ be an eigenvector of signless Laplacian matrix of C_{2n}, where $(u'_i) = \sin\frac{(2i+1)k\pi}{2n}, (i = 0, \dots, 2n-1)$ and $\lambda_k = 2 + 2\cos(\frac{(k+1)\pi}{2n}), (k = 0, \dots, 2n-1)$ be its eigenvalue. We note that $u'_0 = -u'_{2n-1}, u'_1 = -u'_{2n-2}, \dots, u'_{n-1} = -u'_n$. Then we can write the equation $SL(C_{2n})\mathbf{u}' = \lambda_k\mathbf{u}'$ as

$$\begin{pmatrix} 2 & 1 & 0 & \cdots & \cdots & 1 \\ 1 & 2 & 1 & 0 & & 0 \\ 0 & 1 & 2 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & 1 & 2 & 1 \\ 1 & 0 & \cdots & \cdots & 1 & 2 \end{pmatrix} \begin{pmatrix} u'_0 \\ u'_1 \\ \vdots \\ \vdots \\ u'_{2n-2} \\ u'_{2n-1} \end{pmatrix} = \lambda_k\mathbf{u}'.$$

$$\begin{aligned}
2u'_0 + u'_1 - u'_0 &= u'_0 + u'_1 = \lambda_k u'_0, \\
u'_0 + 2u'_1 + u'_2 &= \lambda_k u'_1, \\
&\vdots \qquad (8) \\
u'_{n-2} + 2u'_{n-1} - u'_{n-1} &= u'_{n-2} + u'_{n-1} = \lambda_k u'_{n-1}.
\end{aligned}$$

Comparing Equation 7 and Equation 8, we have $P_n\mathbf{u} = \lambda_k\mathbf{u}$, where $\mathbf{u} = (u'_i), (i = 0, \dots, n-1)$. That is $\lambda_k, (k = 0, \dots, n-1)$ are eigenvalues of P_n and $\mathbf{u}$ is an eigenvector of λ_k. Since $\lambda_i \neq \lambda_j$ for ($i \neq j$ and $0 \leq i, j \leq n-1$), we have n different eigenvalues of P_n and that is the complete set of signless eigenvalues of P_n.

□

4.2. Tridiagonal Matrices

In this section, we derive eigenvalues of adjacency matrices of paths and cycles using Chebyshev polynomials.

Definition 37. Let $T_0(x) = 1$ and $U_0(x) = 0$. For $n \in \mathbf{N}$, $T_n(x)$ and $U_n(x)$ are defined by

$$\begin{pmatrix} T_{n+1}(x) \\ U_{n+1}(x) \end{pmatrix} = \begin{pmatrix} x & x^2-1 \\ 1 & x \end{pmatrix} \begin{pmatrix} T_n(x) \\ U_n(x) \end{pmatrix}.$$

We call $T_n(x)$ as the Chebyshev polynomials of the first kind, and $U_n(x)$ as the Chebyshev polynomials of the second kind.

Example 5. By using the above definition we have,

$$\begin{aligned}
\begin{pmatrix} T_1(x) \\ U_1(x) \end{pmatrix} &= \begin{pmatrix} x & x^2-1 \\ 1 & x \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ 1 \end{pmatrix}, \\
\begin{pmatrix} T_2(x) \\ U_2(x) \end{pmatrix} &= \begin{pmatrix} x & x^2-1 \\ 1 & x \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} 2x^2-1 \\ 2x \end{pmatrix}, \\
\begin{pmatrix} T_3(x) \\ U_3(x) \end{pmatrix} &= \begin{pmatrix} x & x^2-1 \\ 1 & x \end{pmatrix} \begin{pmatrix} 2x^2-1 \\ 2x \end{pmatrix} \\
&= \begin{pmatrix} 4x^3-3x \\ 4x^2-1 \end{pmatrix}.
\end{aligned}$$

Proposition 14. *$T_0(x) = 1$, $T_1(x) = x$, $U_0(x) = 0$, $U_1(x) = 1$,*

$$\begin{aligned}
T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x), \text{ and} \\
U_{n+1}(x) &= 2xU_n(x) - U_{n-1}(x).
\end{aligned}$$

□

Proposition 15.

$$\begin{aligned}
\cos n\theta &= T_n(\cos\theta), \\
\sin n\theta &= U_n(\cos\theta)\sin\theta.
\end{aligned}$$

□

We note that the degree of the polynomial $T_n(x)$ is n and the degree of the polynomial $U_n(x)$ is $n-1$ for $n \geq 2$.

Proposition 16. *Let $x = \cos\theta$ and $n \geq 2$. Then*

$$\begin{aligned}
T_n(x) = 0 \quad &\Leftrightarrow \quad x = \cos(\frac{(2k+1)\pi}{2n}) \ (k = 0, \dots, n-1). \\
U_n(x) = 0 \quad &\Leftrightarrow \quad x = \cos(\frac{k\pi}{n}) \ (k = 1, \dots, n-1).
\end{aligned}$$

□

The determinant of tridiagonal matrices can be represented by using recurrence relations. We consider tridiagonal matrices with similar diagonal elements. Then we derive a formula for eigenvalues of tridiagonal matrices.

Definition 38. A $n \times n$ tridiagonal matrix $A_n = (a_{ij})$ is a matrix which has the form

$$A_n = \begin{pmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0 \\ \gamma_1 & \alpha_2 & \beta_2 & \ddots & \vdots \\ 0 & \gamma_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \alpha_{n-1} & \beta_{n-1} \\ 0 & \cdots & 0 & \gamma_{n-1} & \alpha_n \end{pmatrix}.$$

Proposition 17. *Let $n \geq 2$, $|A_0| = 1$, and $|A_1| = \alpha_1$. Then we have,*

$$|A_n| = \alpha_n |A_{n-1}| - \beta_{n-1}\gamma_{n-1}|A_{n-2}|.$$

Proposition 18. *Eigenvalues of adjacency matrix of a path graph are given by $\lambda_k(A(P_n)) = 2\cos(\frac{k\pi}{n+1})(k = 1, \ldots, n)$.*

Proof. The matrix $\lambda I - P_n$ is a tridiagonal matrix with $\alpha_i = \lambda$, $\beta_i = \gamma_i = -1$. Let $f_n(\lambda) = |\lambda I - P_n|$. By Proposition 17, $f_n(\lambda)$ is defined by $f_n(\lambda) = \lambda f_{n-1}(\lambda) - f_{n-2}(\lambda)$, where $f_0(\lambda) = 1$ and $f_1(\lambda) = \lambda$. Let $g_n(\lambda) = U_{n+1}(\frac{\lambda}{2})$. Then

$$\begin{aligned} g_0(\lambda) &= U_1(\frac{\lambda}{2}) = 1, \\ g_1(\lambda) &= U_2(\frac{\lambda}{2}) = 2(\frac{\lambda}{2}) = \lambda, \\ g_n(\lambda) &= U_{n+1}(\frac{\lambda}{2}) \\ &= 2\frac{\lambda}{2}U_n(\frac{\lambda}{2}) - U_{n-1}(\frac{\lambda}{2}) \ (\textit{by prop.}\ 14) \\ &= \lambda g_{n-1}(\lambda) - g_{n-2}(\lambda). \end{aligned}$$

Then we have $f_n(\lambda) = g_n(\lambda) = U_{n+1}(\frac{\lambda}{2})$. That is

$$\begin{aligned} f_n(\lambda) = 0 \ &\Leftrightarrow \ U_{n+1}(\frac{\lambda}{2}) = 0. \\ &\Leftrightarrow \ \frac{\lambda}{2} = \cos(\frac{k\pi}{n+1}) \ (k = 1, \ldots, n). \\ &\Leftrightarrow \ \lambda = 2\cos(\frac{k\pi}{n+1}) \ (k = 1, \ldots, n). \end{aligned}$$

Thus we obtain the result. □

Proposition 19. *Eigenvalues of adjacency matrix of a cycle are given by $\lambda_k(A(C_n)) = 2\cos(\frac{2k\pi}{n})(k = 1, \ldots, n)$.*

Proof. The matrix $\lambda I - C_n$ is not a tridiagonal matrix. But we have $|\lambda I - C_n| = 2(T_n(\frac{\lambda}{2}) - 1)$. Since $T_n(x) = 1 \Leftrightarrow \cos n\theta = 1 \Leftrightarrow \theta = \frac{2k\pi}{n}$. We obtain

$$|\lambda I - C_n| = \prod_{k=1}^{n}(\lambda - 2\cos(\frac{2k\pi}{n})).$$

□

Proposition 20. *Let $P_n = (V_n, E_n)$ be a path graph. If $G = (V_n, E_n \cup \{(v_1, v_1)\}, \{(v_n, v_n)\})$ then the eigenvalues of Laplacian matrix of G are given by $\lambda_k = a + 2\cos(\frac{k\pi}{n+1})$, $(k = 1, \ldots, n)$.*

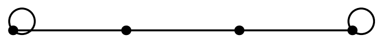


Figure 6: Path graph with equal vertex degrees.

Proof. Let $L(P_n)$ the Laplacian matrix of a path graph with vertex weight a on n vertices. The matrix $\lambda I - L(P_n)$ is a tridiagonal matrix with $\alpha_i = \lambda - a$, $\beta_i = \gamma_i = -1$. Let $f_n(\lambda) = |\lambda I - L(P_n)|$ and $f_n(\lambda)$ is defined by $f_n(\lambda) = \lambda f_{n-1}(\lambda - a) - f_{n-2}(\lambda)$, where $f_0(\lambda) = 1$ and $f_1(\lambda) = \lambda - a$. Let $g_n(\lambda) = U_{n+1}(\frac{\lambda - a}{2})$. Since

$$\begin{aligned} g_0(\lambda) &= U_1(\frac{\lambda - a}{2}) = 1, \\ g_1(\lambda) &= U_2(\frac{\lambda - a}{2}) = 2(\frac{\lambda - a}{2}) = \lambda - a, \textit{and} \\ g_n(\lambda) &= U_{n+1}(\frac{\lambda - a}{2}) \\ &= 2(\frac{\lambda - a}{2})U_n(\frac{\lambda - a}{2}) - U_{n-1}(\frac{\lambda - a}{2}) \\ &= (\lambda - a)g_{n-1}(\lambda) - g_{n-2}(\lambda), \end{aligned}$$

we have $f_n(\lambda) = g_n(\lambda) = U_{n+1}(\frac{\lambda - a}{2})$. That is

$$\begin{aligned} f_n(\lambda) = 0 \ &\Leftrightarrow \ U_{n+1}(\frac{\lambda - a}{2}) = 0 \\ &\Leftrightarrow \ \frac{\lambda - a}{2} = \cos(\frac{k\pi}{n+1}) \ (k = 1, \ldots, n) \\ &\Leftrightarrow \ \lambda = a + 2\cos(\frac{k\pi}{n+1}) \ (k = 1, \ldots, n). \end{aligned}$$

□

4.3. Determinant of tridiagonal matrices

Let $n \geq 2$. We define a $n \times n$ matrix $A_n(a, b)$ as follows:

$$A_n(a,b) = \begin{pmatrix} a & b & 0 & \cdots & \cdots & \cdots & 0 \\ b & a & b & 0 & & & \vdots \\ 0 & b & a & b & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & 0 & b & a & b & 0 \\ \vdots & & & 0 & b & a & b \\ 0 & \cdots & \cdots & \cdots & 0 & b & a \end{pmatrix}.$$

Lemma 16. *Let $n \geq 2$ and $a \neq 2b$.*

$$|A_n(a,b)| = b^n \cdot \frac{\sin(n+1)\theta}{\sin\theta},$$

where $\frac{a}{2b} = \cos\theta$ *($0 < \theta < 2\pi$).*

Proof. By Proposition 17, we have $|A_n(a,1)| = a|A_{n-1}(a,1)| - |A_{n-2}(a,1)|$. Let $|A_0(a,1)| = 1$ and $|A_1(a,1)| = a$. Since $U_1(x) = 1$, $U_2(x) = 2x$ and $U_{n+1} = 2xU_n(x) - U_{n-1}(x)$, we have $|A_n(a,1)| = U_{n+1}\left(\frac{a}{2}\right)$ and $A_n(a,1) = \frac{\sin(n+1)\theta}{\sin\theta}$, where $\frac{a}{2} = \cos\theta$. Since $A_n(a,b) = b^n \cdot A_n\left(\frac{a}{b},1\right)$, we have

$$|A_n(a,b)| = b^n \cdot \left|A_n\left(\frac{a}{b},1\right)\right| = b^n \cdot U_{n+1}\left(\frac{a}{2b}\right) = b^n \frac{\sin(n+1)\alpha}{\sin\alpha}$$

where $\frac{a}{2b} = \cos\alpha$. □

Example 6. **(i)** $\left|A_n\left(\lambda - 1, \frac{1}{2}\right)\right| = \left(\frac{1}{2}\right)^n \frac{\sin(n+1)\alpha}{\sin\alpha}$, where $\lambda = 1 + \cos\alpha$.

(ii) $\left|A_n\left(\eta - \frac{2}{3}, \frac{1}{3}\right)\right| = \left(\frac{1}{3}\right)^n \frac{\sin(n+1)\beta}{\sin\beta}$, where $\eta = \frac{2}{3}(1+\cos\beta)$.

Let $n \geq 3$. We define a $n \times n$ matrix $B_n(a_0, b_0, a, b)$ and $C_n(a, b, a_0, b_0)$ as follows:

$$B_n(a_0,b_0,a,b) = \left(\begin{array}{c|cccc} a_0 & b_0 & 0 & \cdots & 0 \\ \hline b_0 & & & & \\ 0 & & A_{n-1}(a,b) & & \\ \vdots & & & & \\ 0 & & & & \end{array}\right)$$

$$C_n(a,b,a_0,b_0) = \left(\begin{array}{cccc|c} & & & & 0 \\ & A_{n-1}(a,b) & & & \vdots \\ & & & & 0 \\ & & & & b_0 \\ \hline 0 & \cdots & 0 & b_0 & a_0 \end{array}\right)$$

We note that

$$\begin{aligned} |B_n(a_0,b_0,a,b)| &= a_0|A_{n-1}(a,b)| - b_0^2|A_{n-2}(a,b)|, \text{ and} \\ |C_n(a,b,a_0,b_0)| &= a_0|A_{n-1}(a,b)| - b_0^2|A_{n-2}(a,b)|. \end{aligned}$$

We define functions

$$\begin{aligned} g_n(\beta) &= 2\sin((n+1)\beta) + \sin n\beta - \sin((n-1)\beta), \text{ and} \\ h_n(\gamma) &= 2\sin((n+1)\gamma) - \sin n\gamma - \sin((n-1)\gamma) \end{aligned}$$

before introducing the next Lemma.

Lemma 17. *Let $n \geq 3$.*

1. $\left|B_n(\lambda-1, \frac{1}{\sqrt{2}}, \lambda-1, \frac{1}{2})\right| = \frac{1}{2^{n-1}}\cos n\alpha$, *where* $\lambda = 1 + \cos\alpha$.

2. $\left|C_n(\eta-\frac{2}{3}, \frac{1}{3}, \eta-\frac{1}{2}, \frac{1}{\sqrt{6}})\right| = \frac{1}{2\cdot 3^n \cdot \sin\beta} g_n(\beta)$, *where* $\eta = \frac{2}{3}(1+\cos\beta)$.

3. $\left|C_n(\mu-\frac{4}{3}, \frac{1}{3}, \mu-\frac{3}{2}, \frac{1}{\sqrt{6}})\right| = \frac{1}{2\cdot 3^n \cdot \sin\gamma} h_n(\gamma)$, *where* $\mu = \frac{2}{3}(2+\cos\gamma)$.

Proof. 1.

$$\begin{aligned} \left|B_n(\lambda-1, \frac{1}{\sqrt{2}}, \lambda-1, \frac{1}{2})\right| &= (\lambda-1)\left|A_{n-1}(\lambda-1,\frac{1}{2})\right| \\ &\quad -\frac{1}{2}\left|A_{n-2}(\lambda-1,\frac{1}{2})\right| \end{aligned}$$

$$\begin{aligned} &= (\lambda-1)\left(\frac{1}{2}\right)^{n-1}\frac{\sin n\alpha}{\sin\alpha} - \frac{1}{2}\left(\frac{1}{2}\right)^{n-2}\frac{\sin(n-1)\alpha}{\sin\alpha} \\ &= \left(\frac{1}{2}\right)^{n-1}\cdot\frac{1}{\sin\alpha}((\lambda-1)\sin n\alpha - \sin(n-1)\alpha) \\ &= \left(\frac{1}{2}\right)^{n-1}\cdot\frac{1}{\sin\alpha}(\cos\alpha\sin n\alpha - \sin(n\alpha-\alpha)) \\ &= \left(\frac{1}{2}\right)^{n-1}\cdot\frac{1}{\sin\alpha}(\cos n\alpha\sin\alpha) \\ &= \left(\frac{1}{2}\right)^{n-1}\cos n\alpha \end{aligned}$$

2.

$$\begin{aligned} \left|C_n(\eta-\frac{2}{3}, \frac{1}{3}, \eta-\frac{1}{2}, \frac{1}{\sqrt{6}})\right| &= \left(\eta-\frac{1}{2}\right)\left|A_{n-1}\left(\eta-\frac{2}{3},\frac{1}{3}\right)\right| \\ &\quad -\frac{1}{6}\left|A_{n-2}\left(\eta-\frac{2}{3},\frac{1}{3}\right)\right| \end{aligned}$$

$$\begin{aligned} &= \left(\eta-\frac{1}{2}\right)\left(\frac{1}{3}\right)^{n-1}\frac{\sin n\beta}{\sin\beta} - \frac{1}{6}\left(\frac{1}{3}\right)^{n-2}\frac{\sin(n-1)\beta}{\sin\beta} \\ &= \left(\frac{1}{3}\right)^{n-1}\left(\left(\eta-\frac{1}{2}\right)\frac{\sin n\beta}{\sin\beta} - \frac{1}{2}\frac{\sin(n-1)\beta}{\sin\beta}\right) \\ &= \left(\frac{1}{3}\right)^{n-1}\left(\left(\frac{1}{6}+\frac{2}{3}\cos\beta\right)\frac{\sin n\beta}{\sin\beta} - \frac{1}{2}\frac{\sin(n-1)\beta}{\sin\beta}\right) \\ &= \left(\frac{1}{3}\right)^{n-1}\frac{1}{6\sin\beta}(\sin n\beta + 4\cos\beta\sin n\beta - 3\sin(n\beta-\beta)) \\ &= \left(\frac{1}{3}\right)^{n-1}\frac{1}{6\sin\beta}(\sin n\beta + 4\cos\beta\sin n\beta - \sin(n\beta-\beta) \\ &\quad - 2\sin n\beta\cos\beta + 2\cos n\beta\sin\beta) \end{aligned}$$

$$= \left(\frac{1}{3}\right)^{n-1} \frac{1}{6\sin\beta}(\sin n\beta + 2\cos n\beta \sin\beta - \sin(n\beta-\beta) + 2\sin n\beta\cos\beta)$$
$$= \left(\frac{1}{3}\right)^{n-1} \frac{1}{6\sin\beta}(\sin n\beta + 2\sin(n\beta+\beta) - \sin(n\beta-\beta))$$
$$= \left(\frac{1}{3}\right)^{n-1} \frac{1}{6\sin\beta}(2\sin(n+1)\beta + \sin n\beta - \sin(n-1)\beta)$$
$$= \left(\frac{1}{3}\right)^{n} \frac{1}{2\sin\beta} g_n(\beta)$$

3.

$$\left|C_n(\mu - \frac{4}{3}, \frac{1}{3}, \mu - \frac{3}{2}, \frac{1}{\sqrt{6}})\right| = \left(\mu - \frac{3}{2}\right)\left|A_{n-1}\left(\mu - \frac{4}{3}, \frac{1}{3}\right)\right| - \frac{1}{6}\left|A_{n-2}\left(\mu - \frac{4}{3}, \frac{1}{3}\right)\right|$$

$$= \left(\mu - \frac{3}{2}\right)\left(\frac{1}{3}\right)^{n-1} \frac{\sin n\gamma}{\sin\gamma} - \frac{1}{6}\left(\frac{1}{3}\right)^{n-2} \frac{\sin(n-1)\gamma}{\sin\gamma}$$
$$= \left(\frac{1}{3}\right)^{n-1}\left(\left(\mu - \frac{3}{2}\right)\frac{\sin n\gamma}{\sin\gamma} - \frac{1}{2}\frac{\sin(n-1)\gamma}{\sin\gamma}\right)$$
$$= \left(\frac{1}{3}\right)^{n-1}\left(\left(-\frac{1}{6} + \frac{2}{3}\cos\gamma\right)\frac{\sin n\gamma}{\sin\gamma} - \frac{1}{2}\frac{\sin(n-1)\gamma}{\sin\gamma}\right)$$
$$= \left(\frac{1}{3}\right)^{n-1} \frac{1}{6\sin\gamma}(-\sin n\gamma + 4\cos\gamma\sin n\gamma - 3\sin(n\gamma-\gamma))$$
$$= \left(\frac{1}{3}\right)^{n-1} \frac{1}{6\sin\gamma}(2\sin(n+1)\gamma - \sin n\gamma - \sin(n-1)\gamma)$$
$$= \left(\frac{1}{3}\right)^{n} \frac{1}{2\sin\gamma} h_n(\gamma)$$

□

4.4. Eigenvalues of $\mathcal{L}(P_n)$

Example 7. The adjacency matrix and the normalized Laplacian matrix of a path graph P_5.

$$A(P_5) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathcal{L}(P_5) = \begin{pmatrix} 1 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 1 \end{pmatrix}$$

Let $n \geq 4$. We define $n \times n$ matrix $Q_n(a_0, b_0, a, b)$ as the following.

$$Q_n(a_0, b_0, a, b) = \left(\begin{array}{c|ccc|c} a_0 & b_0 & 0 & \cdots & 0 \\ \hline b_0 & & & & \vdots \\ 0 & & A_{n-2}(a,b) & & 0 \\ \vdots & & & & b_0 \\ \hline 0 & \cdots & 0 & b_0 & a_0 \end{array}\right)$$

We note that

$$|Q_n(a_0, b_0, a, b)| = a_0|C_{n-1}(a, b, a_0, b_0)| - b_0^2|C_{n-2}(a, b, a_0, b_0)|.$$

Proposition 21. *Let $n \geq 4$. The characteristic polynomial of $\mathcal{L}(P_n)$ is*

$$|\lambda I_n - \mathcal{L}(P_n)| = -\left(\frac{1}{2}\right)^{n-2}(\sin\alpha\sin((n-1)\alpha)),$$

where $\lambda = 1 + \cos\alpha$. That is $\lambda = 1 - \cos(\frac{k\pi}{n-1})$ $(k = 0, \ldots, n-1)$.

Proof. First, we note $\mathcal{L}(P_n) = Q_n\left(1, -\frac{1}{\sqrt{2}}, 1, -\frac{1}{2}\right)$ and $|\lambda I_n - \mathcal{L}(P_n)| = \left|Q_n\left(\lambda - 1, \frac{1}{\sqrt{2}}, \lambda - 1, \frac{1}{2}\right)\right|$.

$$\left|Q_n\left(\lambda - 1, \frac{1}{\sqrt{2}}, \lambda - 1, \frac{1}{2}\right)\right|$$
$$= (\lambda - 1)\left|C_{n-1}\left(\lambda - 1, \frac{1}{2}, \lambda - 1, \frac{1}{\sqrt{2}}\right)\right| - \frac{1}{2}\left|C_{n-2}\left(\lambda - 1, \frac{1}{2}, \lambda - 1, \frac{1}{\sqrt{2}}\right)\right|$$
$$= (\lambda - 1)\left|B_{n-1}\left(\lambda - 1, \frac{1}{\sqrt{2}}, \lambda - 1, \frac{1}{2}\right)\right| - \frac{1}{2}\left|B_{n-2}\left(\lambda - 1, \frac{1}{\sqrt{2}}, \lambda - 1, \frac{1}{2}\right)\right|$$
$$= (\lambda - 1)\left(\frac{1}{2}\right)^{n-2}\cos((n-1)\alpha) - \frac{1}{2}\left(\frac{1}{2}\right)^{n-3}\cos((n-2)\alpha)$$
$$= \left(\frac{1}{2}\right)^{n-2}(\cos\alpha \cdot \cos((n-1)\alpha) - \cos((n-2)\alpha))$$
$$= \left(\frac{1}{2}\right)^{n-2}(\cos\alpha \cdot \cos((n-1)\alpha) - (\cos((n-1)\alpha)\cdot\cos\alpha + \sin((n-1)\alpha)\cdot\sin\alpha))$$
$$= -\left(\frac{1}{2}\right)^{n-2}\sin((n-1)\alpha)\cdot\sin\alpha.$$

We have $\alpha = \dfrac{k\pi}{n-1}$ $(k = 0, \ldots, n-1)$. Since $\cos\left(\dfrac{k\pi}{n-1}\right) = \cos\left(\dfrac{(k+(n-1))\pi}{n-1}\right)$, we have $\lambda = 1 + \cos\left(\dfrac{k\pi}{n-1}\right)$ $(k = 0, \ldots, n-1)$. The set is equal to $\lambda = 1 - \cos\left(\dfrac{k\pi}{n-1}\right)$ $(k = 0, \ldots, n-1)$. □

4.5. Eigenvalues of weighted paths and $\mathcal{L}(R_{n,k})$

Example 8. The adjacency matrix and the normalized Laplacian matrix of a weighted path graph $P_{4,3}$.

$$A(P_{4,3}) = \begin{pmatrix} 0&1&0&0&0&0&0\\ 1&0&1&0&0&0&0\\ 0&1&0&1&0&0&0\\ 0&0&1&0&1&0&0\\ 0&0&0&1&1&1&0\\ 0&0&0&0&1&1&1\\ 0&0&0&0&0&1&1 \end{pmatrix}$$

$$\mathcal{L}(P_{4,3}) = \begin{pmatrix} 1 & -\frac{1}{\sqrt{2}} & 0&0&0&0&0\\ -\frac{1}{\sqrt{2}} & 1 & -\frac12 & 0&0&0&0\\ 0 & -\frac12 & 1 & -\frac12 & 0&0&0\\ 0&0&-\frac12&1&-\frac{1}{\sqrt6}&0&0\\ 0&0&0&-\frac{1}{\sqrt6}&\frac23&-\frac13&0\\ 0&0&0&0&-\frac13&\frac23&-\frac{1}{\sqrt6}\\ 0&0&0&0&0&-\frac{1}{\sqrt6}&\frac12 \end{pmatrix}$$

Let $n \ge 3$ and $k \ge 3$. Then

$$\mathcal{L}(P_{n,k}) = \left(\begin{array}{c|c} B_n(1, -\frac{1}{\sqrt2}, 1, -\frac12) & X_{n,k} \\ \hline X^t_{n,k} & C_k(\frac23, -\frac13, \frac12, -\frac{1}{\sqrt6}) \end{array}\right),$$

where $X_{n,k}$ is the $n \times k$ matrix defined by

$$X_{n,k} = \begin{pmatrix} 0 & \cdots & \cdots & 0\\ \vdots & & & \vdots\\ 0 & 0 & & \vdots \\ -\frac{1}{\sqrt6} & 0 & \cdots & 0 \end{pmatrix}.$$

Theorem 5. *Let $n \ge 3$ and $k \ge 3$. The characteristic polynomial of $\mathcal{L}(P_{n,k})$ is*

$$\left|\lambda I_{n+k} - \mathcal{L}(P_{n,k})\right| = p_{n,k}(\lambda),$$

where

$$p_{n,k}(\lambda) = \frac{1}{2^n 3^k \sin\beta}(g_k(\beta)\cos(n\alpha)) - g_{k-1}(\beta)\cos((n-1)\alpha)),$$

$\lambda = 1 + \cos\alpha$ and $\lambda = \frac23(1+\cos\beta)$.

Proof. Since $|B_n(\lambda-1, \frac{1}{\sqrt2}, \lambda-1, \frac12)| = \frac{\cos(n\alpha)}{2^{n-1}}$ and $|C_k(\lambda - \frac23, \frac13, \lambda-\frac12, \frac{1}{\sqrt6})| = \frac{g_k(\beta)}{2\cdot 3^k\cdot\sin\beta}$, we have

$$\left|\lambda I_{n+k} - \mathcal{L}(P_{n,k})\right| = \left|\begin{array}{c|c} B_n(\lambda-1, \frac{1}{\sqrt2}, \lambda-1, \frac12) & X_{n,k} \\ \hline X^t_{n,k} & C_k(\lambda-\frac23, \frac13, \lambda-\frac12, \frac{1}{\sqrt6}) \end{array}\right|$$

$$\begin{aligned}
&= -\frac14|B_{n-2}(\lambda-1, \tfrac{1}{\sqrt2}, \lambda-1, \tfrac12)|\cdot|C_k(\lambda-\tfrac23, \tfrac13, \lambda-\tfrac12, \tfrac{1}{\sqrt6})|\\
&\quad +(\lambda-1)|B_{n-1}(\lambda-1, \tfrac{1}{\sqrt2}, \lambda-1, \tfrac12)|\cdot|C_k(\lambda-\tfrac23, \tfrac13, \lambda-\tfrac12, \tfrac{1}{\sqrt6})|\\
&\quad -\frac16|B_{n-1}(\lambda-1, \tfrac{1}{\sqrt2}, \lambda-1, \tfrac12)|\cdot|C_{k-1}(\lambda-\tfrac23, \tfrac13, \lambda-\tfrac12, \tfrac{1}{\sqrt6})|\\
&= -\frac14\cdot\frac{\cos((n-2)\alpha)}{2^{n-3}}\cdot\frac{g_k(\beta)}{2\cdot3^k\cdot\sin\beta} + \cos\alpha\cdot\frac{\cos((n-1)\alpha)}{2^{n-2}}\cdot\\
&\quad \frac{g_k(\beta)}{2\cdot3^k\cdot\sin\beta} - \frac16\frac{\cos((n-1)\alpha)}{2^{n-2}}\cdot\frac{g_{k-1}(\beta)}{2\cdot3^{k-1}\cdot\sin\beta}\\
&= \frac{1}{2^n\cdot3^k\cdot\sin\beta}(-\cos((n-2)\alpha)g_k(\beta) + 2\cos\alpha\cos((n-1)\alpha)g_k(\beta)\\
&\quad -\cos((n-1)\alpha)g_{k-1}(\beta))\\
&= \frac{1}{2^n\cdot3^k\cdot\sin\beta}(\cos(n\alpha)g_k(\beta) - \cos((n-1)\alpha)g_{k-1}(\beta))\\
&= p_{n,k}(\lambda)
\end{aligned}$$

We note that

$$\begin{aligned}
& -\cos((n-2)\alpha) + 2\cos\alpha\cos((n-1)\alpha)\\
=\ & -\cos\alpha\cos((n-1)\alpha) - \sin\alpha\sin((n-1)\alpha)\\
& +2\cos\alpha\cos((n-1)\alpha)\\
=\ & \cos\alpha\cos((n-1)\alpha) - \sin\alpha\sin((n-1)\alpha)\\
=\ & \cos(n\alpha).
\end{aligned}$$

□

Lemma 18. *Let $k \ge 3$.*

1. *If $\frac{(4k-2)\pi}{4k-1} < \alpha < \pi$, $0 < \beta < \pi$ and $1 + \cos\alpha = \frac23(1 + \cos\beta)$ then $\frac{(2k+1)\pi}{2(k+1)} < \beta$.*
2. *If $\frac{(2k+1)\pi}{2(k+1)} < \beta < \pi$ then $g_k(\beta) \ne 0$ and $\frac{g_{k-1}(\beta)}{g_k(\beta)} < -1$.*
3. *If $n = 2k$ ($k \ge 0$), $\frac{(2n-2)\pi}{2n-1} < \alpha < \pi$, $0 < \beta < \pi$ and $1 + \cos\alpha = \frac23(1+\cos\beta)$ then $g_k(\beta)\cos(n\alpha) - g_{k-1}(\beta)\cos((n-1)\alpha) \ne 0$.*

Proof. 1. Since $k \ge 3$, we have $\frac{4k-1}{k+1} = 4 - \frac{5}{k+1} \ge \frac{11}{4}$ and $\frac{1}{4k-1} \le \frac{4}{11(k+1)}$.

Since $\frac{33}{8}\sqrt{\frac23} > 3.36 > \pi$, we have $\sqrt{\frac32}\cdot\frac12\cdot\frac{4\pi}{11} < \sqrt{\frac32}\cdot\frac12\cdot\frac{4}{11}\cdot\frac{33}{8}\sqrt{\frac23} = \frac34$.

Since $\frac{3x}{2} \le \sin(\frac{\pi x}{2})$ $(0 \le x \le \frac13)$, we have $\frac{3}{4(k+1)} \le \sin\frac{\pi}{4(k+1)}$. Since $1 + \cos\alpha = \frac23(1+\cos\beta)$, we have $\cos^2\frac{\alpha}{2} = \frac23\cos^2\frac{\beta}{2}$ and $\cos\frac{\beta}{2} = \sqrt{\frac32}\cos\frac{\alpha}{2}$.

$$
\begin{aligned}
\sin(\frac{\pi-\beta}{2}) = \cos\frac{\beta}{2} &= \sqrt{\frac{3}{2}}\cos\frac{\alpha}{2} \\
&= \sqrt{\frac{3}{2}}\sin(\frac{\pi-\alpha}{2}) \\
&\le \sqrt{\frac{3}{2}}(\frac{\pi-\alpha}{2}) \\
&< \sqrt{\frac{3}{2}}\cdot\frac{1}{2}\cdot(\frac{\pi}{4k-1}) \\
&< \sqrt{\frac{3}{2}}\cdot\frac{1}{2}\cdot(\frac{4\pi}{11(k+1)} \\
&< \frac{3}{4(k+1)} \\
&= \frac{1}{2}\cdot\frac{3}{2(k+1)} \\
&\le \sin\frac{\pi}{4(k+1)}
\end{aligned}
$$

Then $\frac{\pi-\beta}{2} < \frac{\pi}{4(k+1)}$ and $\frac{(2k+1)\pi}{2(k+1)} < \beta$.

2. Let $\beta' = \pi - \beta$. Then $0 < \beta' < \frac{\pi}{2(k+1)}$. We note that if k is even then $\sin(k\beta) = -\sin(k\beta')$ and if k is odd then $\sin(k\beta) = \sin(k\beta')$. Since $y = \sin x$ is convex on $0 < x < \frac{\pi}{2}$, $\sin(tx_1 + (1-t)x_2) > t\sin x_1 + (1-t)\sin x_2$ for $0 < x_1 < x_2 < \frac{\pi}{2}$ and $0 < t < 1$. Since $0 < (k-2)\beta' < k\beta' < (k+1)\beta' < \frac{\pi}{2}$ and $\frac{1}{3}(k-2) + (1-\frac{1}{3})(k+1) = k$, we have $\sin(k\beta') > \frac{1}{3}\sin((k-2)\beta') + \frac{2}{3}\sin((k+1)\beta')$.

$$
\begin{aligned}
g_{k-1}(\beta) + g_k(\beta) &= 2\sin(k\beta) + \sin((k-1)\beta) - \sin((k-2)\beta) \\
&\quad +2\sin((k+1)\beta) + \sin(k\beta) - \sin((k-1)\beta) \\
&= -\sin((k-2)\beta) + 3\sin(k\beta) + 2\sin((k+1)\beta)
\end{aligned}
$$

If k is even then $g_k(\beta) = 2\sin((k+1)\beta) + \sin(k\beta) - \sin((k-1)\beta)$ $= 2\sin((k+1)\beta') - \sin(k\beta') - \sin((k-1)\beta) > 0$.

$$
\begin{aligned}
g_{k-1}(\beta) + g_k(\beta) &= -\sin((k-2)\beta) + 3\sin(k\beta) + 2\sin((k+1)\beta) \\
&= \sin((k-2)\beta) - 3\sin(k\beta) + 2\sin((k+1)\beta) \\
&= 3(\frac{1}{3}\sin((k-2)\beta') + \frac{2}{3}\sin((k+1)\beta') \\
&\quad -\sin(k\beta')) < 0.
\end{aligned}
$$

Since $g_k(\beta) > 0$, $\frac{g_{k-1}(\beta)}{g_k(\beta)} + 1 < 0$.

If k is odd then $g_k(\beta) = 2\sin((k+1)\beta) + \sin(k\beta) - \sin((k-1)\beta)$ $= -2\sin((k+1)\beta') + \sin(k\beta') + \sin((k-1)\beta) > 0$.

$$
\begin{aligned}
g_{k-1}(\beta) + g_k(\beta) &= -\sin((k-2)\beta) + 3\sin(k\beta) + 2\sin((k+1)\beta) \\
&= -\sin((k-2)\beta) + 3\sin(k\beta) - 2\sin((k+1)\beta) \\
&= 3(\sin(k\beta')) - \frac{1}{3}\sin((k-2)\beta') \\
&\quad -\frac{2}{3}\sin((k+1)\beta') > 0.
\end{aligned}
$$

Since $g_k(\beta) < 0$, $\frac{g_{k-1}(\beta)}{g_k(\beta)} + 1 < 0$.

3. Let $\alpha' = \pi - \alpha$. Then $0 < \alpha' < \frac{\pi}{4k-1}$. We note that if n is even then $\cos(n\alpha) = \cos(n\alpha')$ and if n is odd then $\cos(n\alpha) = -\cos(n\alpha')$.

$$
\begin{aligned}
g_k(\beta)\cos(n\alpha) - g_{k-1}(\beta)\cos((n-1)\alpha) &= g_k(\beta)\cos((n-1)\alpha) \\
&\quad (\frac{\cos(n\alpha)}{\cos((n-1)\alpha)} - \frac{g_{k-1}(\beta)}{g_k(\beta)}).
\end{aligned}
$$

Since $n = 2k$, we have $\cos(n\alpha) = \cos(n\alpha')$ and $\cos((n-1)\alpha) = -\cos((n-1)\alpha') < 0$. Since $0 < (n-1)\alpha' < n\alpha' < \pi$ and $(n-1)\alpha' < \frac{(n-1)\pi}{2n-1} < \frac{\pi}{2}$, we have $\cos((n-1)\alpha') > \cos(n\alpha')$, $-\cos((n-1)\alpha) > \cos(n\alpha)$ and $\frac{\cos(n\alpha)}{\cos((n-1)\alpha)} > -1$. Since $\frac{g_{k-1}(\beta)}{g_k(\beta)} < -1$, we have $\frac{\cos(n\alpha)}{\cos((n-1)\alpha)} - \frac{g_{k-1}(\beta)}{g_k(\beta)} > 0$.

If k is even then $g_k(\beta) > 0$ and $\cos((n-1)\alpha) < 0$, then

$$g_k(\beta)\cos(n\alpha) - g_{k-1}(\beta)\cos((n-1)\beta) < 0.$$

If k is odd then $g_k(\beta) < 0$ and $\cos((n-1)\alpha) < 0$, then

$$g_k(\beta)\cos(n\alpha) - g_{k-1}(\beta)\cos((n-1)\beta) > 0.$$

□

Proposition 22. *If $k \ge 3$ and $\lambda_2(\mathcal{L}(P_{2k,k}))$ the second eigenvalue of $\mathcal{L}(P_{2k,k})$ then*

$$1 - \cos\frac{\pi}{4k-1} \le \lambda_2(\mathcal{L}(P_{2k,k})).$$

Proof. Let $0 < \alpha < \pi$. If $1+\cos\alpha < 1-\cos\frac{\pi}{4k-1}$ then $\cos\alpha < -\cos\frac{\pi}{4k-1}$ and $\frac{\pi}{4k-1} < \alpha$. By Theorem 5 and Lemma 18, we have $p_{n,k}(\lambda) \ne 0$ if $\frac{\pi}{4k-1} < \alpha < \pi$ and $\lambda = 1 + \cos\alpha$. This shows that

$$1 - \cos\frac{\pi}{4k-1} \le \lambda_2(\mathcal{L}(P_{2k,k})).$$

□

Example 9. If $k = 3$ then $n = 6$ and $1-\cos\frac{\pi}{4k-1} = 0.0405\cdots$. If $k = 4$ then $n = 8$ and $1-\cos\frac{\pi}{4k-1} = 0.02185\cdots$. The blue curve in the Figure 7 is $y = p_{6,3}(x)$ and the red curve is $y = p_{8,4}(x)$.

4.6. Eigenvalues of $\mathcal{L}(R_{n,k})$

Example 10. The adjacency matrix and the normalized Laplacian matrix of a graph $R_{5,5}$.

$$
A(R_{5,5}) = \begin{pmatrix}
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&1&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&1&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&1\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0
\end{pmatrix}
$$

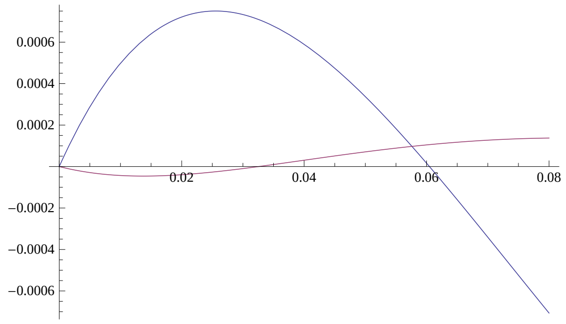


Figure 7: Eigenvalues of $P_{6,3}$ and $P_{8,4}$

$\mathcal{L}(R_{5,5})$ can be written as

$$\left(\begin{array}{cccccccccccccccccccc}
1 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{\sqrt{2}} & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 1 & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 1 & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 1 & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{6}} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 1 & -\frac{1}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 1 & -\frac{1}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 1 & -\frac{1}{\sqrt{6}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{6}} & 1
\end{array}\right)$$

Theorem 6. *Let $n \geq 3$, $k \geq 3$. The characteristic polynomial of $\mathcal{L}(R_{n,k})$ is*

$$|\lambda I_{2(n+k)} - \mathcal{L}(R_{n,k})| = p_{n,k}(\lambda) \cdot q_{n,k}(\lambda),$$

where

$$\begin{array}{rcl} p_{n,k}(\lambda) & = & \dfrac{1}{2^n 3^k \sin\beta}(g_k(\beta)\cos(n\alpha)) - g_{k-1}(\beta)\cos((n-1)\alpha)), \\ q_{n,k}(\lambda) & = & \dfrac{1}{2^n 3^k \sin\gamma}(h_k(\gamma)\cos(n\alpha) - h_{k-1}(\gamma)\cos((n-1)\alpha)), \end{array}$$

and $\lambda = 1 + \cos\alpha = \dfrac{2}{3}(1 + \cos\beta) = \dfrac{2}{3}(2 + \cos\gamma)$.

Proof. Since $|B_n(\lambda - 1, \dfrac{1}{\sqrt{2}}, \lambda - 1, \dfrac{1}{2})| = \dfrac{\cos(n\alpha)}{2^{n-1}}$ and $|C_k(\lambda - \dfrac{4}{3}, \dfrac{1}{3}, \lambda - \dfrac{3}{2}, \dfrac{1}{\sqrt{6}})| = \dfrac{h_k(\gamma)}{2 \cdot 3^k \cdot \sin\gamma}$, we have

$$\begin{array}{rl} & \left|\begin{array}{c|c} B_n(\lambda-1, \frac{1}{\sqrt{2}}, \lambda-1, \frac{1}{2}) & X_{n,k} \\ \hline X_{n,k}^t & C_k(\lambda - \frac{4}{3}, \frac{1}{3}, \lambda - \frac{3}{2}, \frac{1}{\sqrt{6}}) \end{array}\right| \\ = & -\dfrac{1}{4}|B_{n-2}(\lambda - 1, \dfrac{1}{\sqrt{2}}, \lambda - 1, \dfrac{1}{2})| \cdot |C_k(\lambda - \dfrac{4}{3}, \dfrac{1}{3}, \lambda - \dfrac{3}{2}, \dfrac{1}{\sqrt{6}})| \\ & +(\lambda - 1)|B_{n-1}(\lambda - 1, \dfrac{1}{\sqrt{2}}, \lambda - 1, \dfrac{1}{2})| \cdot |C_k(\lambda - \dfrac{4}{3}, \dfrac{1}{3}, \lambda - \dfrac{3}{2}, \dfrac{1}{\sqrt{6}})| \\ & -\dfrac{1}{6}|B_{n-1}(\lambda - 1, \dfrac{1}{\sqrt{2}}, \lambda - 1, \dfrac{1}{2})| \cdot |C_{k-1}(\lambda - \dfrac{4}{3}, \dfrac{1}{3}, \lambda - \dfrac{3}{2}, \dfrac{1}{\sqrt{6}})| \\ = & -\dfrac{1}{4} \cdot \dfrac{\cos((n-2)\alpha)}{2^{n-3}} \cdot \dfrac{h_k(\gamma)}{2 \cdot 3^k \cdot \sin\gamma} \\ & + \cos\alpha \cdot \dfrac{\cos((n-1)\alpha)}{2^{n-2}} \cdot \dfrac{h_k(\gamma)}{2 \cdot 3^k \cdot \sin\gamma} \\ & -\dfrac{1}{6}\dfrac{\cos((n-1)\alpha)}{2^{n-2}} \cdot \dfrac{h_{k-1}(\gamma)}{2 \cdot 3^{k-1} \cdot \sin\gamma} \\ = & \dfrac{1}{2^n \cdot 3^k \cdot \sin\gamma}(-\cos((n-2)\alpha)h_k(\gamma) + 2\cos\alpha\cos((n-1)\alpha) \\ & h_k(\gamma) - \cos((n-1)\alpha)h_{k-1}(\gamma)) \\ = & \dfrac{1}{2^n \cdot 3^k \cdot \sin\gamma}(\cos(n\alpha)h_k(\gamma) - \cos((n-1)\alpha)h_{k-1}(\gamma)) \\ = & q_{n,k}(\lambda). \end{array}$$

We note that

$$\begin{array}{rl} & -\cos((n-2)\alpha) + 2\cos\alpha\cos((n-1)\alpha) \\ = & -\cos\alpha\cos((n-1)\alpha) - \sin\alpha\sin((n-1)\alpha) \\ & +2\cos\alpha\cos((n-1)\alpha) \\ = & \cos\alpha\cos((n-1)\alpha) - \sin\alpha\sin((n-1)\alpha) \\ = & \cos(n\alpha). \end{array}$$

So we have,

$$\begin{array}{rl} & \left|\lambda I_{2(n+k)} - \mathcal{L}(R_{n,k})\right| \\ = & \left|\begin{array}{c|c} B_n(\lambda-1, \frac{1}{\sqrt{2}}, \lambda-1, \frac{1}{2}) & X_{n,k} \\ \hline X_{n,k}^t & C_k(\lambda - \frac{2}{3}, \frac{1}{3}, \lambda - \frac{1}{2}, \frac{1}{\sqrt{6}}) \end{array}\right| \\ & \times\left|\begin{array}{c|c} B_n(\lambda-1, \frac{1}{\sqrt{2}}, \lambda-1, \frac{1}{2}) & X_{n,k} \\ \hline X_{n,k}^t & C_k(\lambda - \frac{4}{3}, \frac{1}{3}, \lambda - \frac{3}{2}, \frac{1}{\sqrt{6}}) \end{array}\right| \\ = & p_{n,k}(\lambda) \times q_{n,k}(\lambda), \end{array}$$

where $\lambda = 1 + \cos\alpha = \dfrac{2}{3}(1 + \cos\beta) = \dfrac{2}{3}(2 + \cos\gamma)$. □

Definition 39. Let $n \geq 1$. we define two matrices $T((a_i)_{1 \leq i \leq n}, (b_i)_{1 \leq i \leq n-1}, (c_i)_{2 \leq i \leq n})$ and F as follows:

$$\begin{array}{rl} & T((a_i)_{1 \leq i \leq n}, (b_i)_{1 \leq i \leq n-1}, (c_i)_{2 \leq i \leq n}) \\ = & \left(\begin{array}{ccccccc} a_1 & b_1 & 0 & 0 & ... & 0 & 0 \\ c_2 & a_2 & b_2 & 0 & ... & 0 & 0 \\ 0 & c_3 & a_3 & b_3 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & ... & 0 & c_{n-2} & a_{n-2} & b_{n-2} & 0 \\ 0 & ... & 0 & 0 & c_{n-1} & a_{n-1} & b_{n-1} \\ 0 & ... & 0 & 0 & 0 & c_n & a_n \end{array}\right), \text{ and} \end{array}$$

$$F = (f_{ij})_{1 \leq i,j \leq n}, \text{ where } f_{ij} = \begin{cases} (-1)^i & (i = j), \\ 0 & (\text{otherwise}). \end{cases}$$

Lemma 19.

$$F^{-1} \cdot T((a_i), (b_i), (c_i)) \cdot F = T((a_i), (-b_i), (-c_i)).$$

Proof. First, we note that $F^{-1} = F$. Each element of b_i or c_i is in odd row and even column or even row and odd column. The right multiplication of F changes the sign of an odd row and the left multiplication of F changes the sing of an odd column. The sign of a_i is changed twice and the sign of b_i or c_i is changed once. So we have $F^{-1} \cdot T((a_i), (b_i), (c_i)) \cdot F = T((a_i), (-b_i), (-c_i))$. □

Proposition 23. *Let $n \geq 1$, $k \geq 2$,*

$$P = \left(\begin{array}{c|c} B_n(1, -\frac{1}{\sqrt{2}}, 1, -\frac{1}{2}) & X_{n,k} \\ \hline X_{n,k}^t & C_k(\frac{2}{3}, -\frac{1}{3}, \frac{1}{2}, -\frac{1}{\sqrt{6}}) \end{array} \right) \text{ and}$$

$$Q = \left(\begin{array}{c|c} B_n(1, -\frac{1}{\sqrt{2}}, 1, -\frac{1}{2}) & X_{n,k} \\ \hline X_{n,k}^t & C_k(\frac{4}{3}, -\frac{1}{3}, \frac{3}{2}, -\frac{1}{\sqrt{6}}) \end{array} \right).$$

1. *Let $\lambda \in \Re$ and $u \in R^{n+k}$. Then $Pu = \lambda u$ if and only if $Q(Fu) = (2 - \lambda)(Fu)$.*
2. *An eigenvalue $\lambda \neq 0$ of P is simple.*
3. *An eigenvalue $\lambda \neq 0$ of Q is simple.*
4. *Let $\lambda \in \Re$, $u = (u_i)_{1 \leq i \leq 2(n+k)} \in R^{2(n+k)}$ and $u_i = u_{n+k+i}$ $(1 \leq i \leq n+k)$. Then $\mathcal{L}(R_{n,k})u = \lambda u$ if and only if $Pu = \lambda u$, where $u = (u_i)_{1 \leq i \leq n+k}$.*
5. *Let $\lambda \in \Re$, $u = (u_i)_{1 \leq i \leq 2(n+k)} \in R^{2(n+k)}$ and $u_i = -u_{n+k+i}$ $(1 \leq i \leq n+k)$. Then $\mathcal{L}(R_{n,k})u = \lambda u$ if and only if $Qu = \lambda u$ where $u = (u_i)_{1 \leq i \leq n+k}$.*

Proof.
1. First, we note that $Q = F^{-1}(2I - P)F$ by Lemma 19. So Q and $2I - P$ have same eigenvalues and Fu is an eigenvector of Q if and only if u is an eigenvector of P.
2. If λ is not simple, we can have an eigenvector $u = (u_i)_{1 \leq i \leq n+k}$, where $u_1 = 0$. By $Pu = \lambda u$ and $\lambda \neq 0$, we have $u = 0$ and it contradict that u is an eigenvector of P. So we have $\lambda (\neq 0)$ is simple.
3. It is similar to 2.
4. Assume $\mathcal{L}(R_{n,k})u = \lambda u$, then we have $Pu = \lambda u$ by direct computations. The converse is also hold.
5. It is similar to 4.

□

Example 11. Let $n = k = 2$. Then

$$\mathcal{L}(R_{2,2}) = \begin{pmatrix} 1 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 1 & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{\sqrt{6}} & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{6}} & 1 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 1 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{\sqrt{6}} \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{\sqrt{6}} & 1 \end{pmatrix},$$

$$P = \begin{pmatrix} 1 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 1 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & -\frac{1}{\sqrt{6}} & \frac{2}{3} & -\frac{1}{\sqrt{6}} \\ 0 & 0 & -\frac{1}{\sqrt{6}} & \frac{1}{2} \end{pmatrix}, \text{ and}$$

$$Q = \begin{pmatrix} 1 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 1 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & -\frac{1}{\sqrt{6}} & \frac{4}{3} & -\frac{1}{\sqrt{6}} \\ 0 & 0 & -\frac{1}{\sqrt{6}} & \frac{3}{2} \end{pmatrix}.$$

Eigenvalues of $R_{2,2}$ are 2., 1.79533, 1.62867, 1, 1, 0.371333, 0.204666 and 0. Corresponding eigenvectors are

$$\begin{pmatrix} 0.707107 & -1. & 1.22474 & -1. & -0.707107 & 1. & -1.22474 & 1. \\ -6.90985 & 7.772 & -3.17291 & 1. & -6.90985 & 7.772 & -3.17291 & 1. \\ -0.868326 & 0.772002 & 0.315168 & -1. & 0.868326 & -0.772002 & -0.315168 & 1. \\ 0.707107 & 0 & -1.22474 & 1 & 0.707104 & 0 & -1.22474 & 1 \\ -0.707107 & 0 & 1.22474 & 1 & 0.707104 & 0 & -1.22474 & -1 \\ -0.868326 & -0.772002 & 0.315168 & 1. & -0.868326 & -0.772002 & 0.315168 & 1. \\ -6.90985 & -7.772 & -3.17291 & -1. & 6.90985 & 7.772 & 3.17291 & 1. \\ 0.707107 & 1. & 1.22474 & 1. & 0.707107 & 1. & 1.22474 & 1. \end{pmatrix}.$$

Eigenvalues of P are 1.79533, 1, 0.371333, and 0. Corresponding eigenvectors are

$$\begin{pmatrix} -6.90985 & 7.772 & -3.17291 & 1. \\ 0.707107 & 0. & -1.22474 & 1. \\ -0.868326 & -0.772002 & 0.315168 & 1. \\ 0.707107 & 1. & 1.22474 & 1. \end{pmatrix}.$$

Eigenvalues of Q are 2, 1.62867, 1, and 0.204666. Corresponding eigenvectors are

$$\begin{pmatrix} -0.707107 & 1. & -1.22474 & 1. \\ 0.868326 & -0.772002 & -0.315168 & 1. \\ -0.707107 & 0. & 1.22474 & 1. \\ 6.90985 & 7.772 & 3.17291 & 1. \end{pmatrix}.$$

Each eigenvector of P is corresponding to an even eigenvector of $\mathcal{L}(R_{2,2})$ and Q an odd eigenvector of $\mathcal{L}(R_{2,2})$. Even though eigenvalues of P and Q are simple, an eigenvalue 1 of $\mathcal{L}(R_{2,2})$ is not simple.

5. COUNTER EXAMPLES FOR $Mcut(G) \neq Lcut(G)$

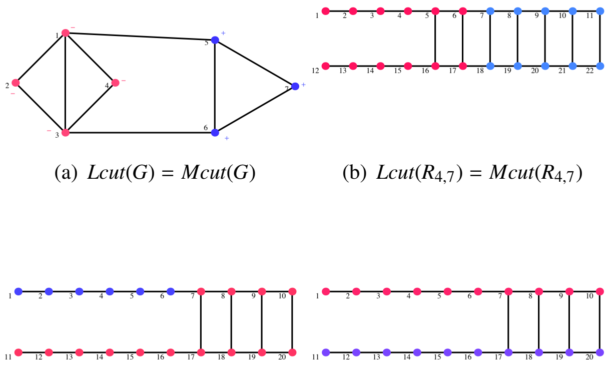
This section present counter example graphs, on which spectral methods and minimum normalized cut produce different clusters.

5.1. $Mcut(G)$ AND $Lcut(G)$

Definition 40 ($Lcut(G)$)**.** Let $G = (V, E)$ be a connected graph, λ_2 the second smallest eigenvalue of $\mathcal{L}(G)$, $U_2 = ((U_2)_i)$ $(1 \leq i \leq |V|)$ a second eigenvector of $\mathcal{L}(G)$ with λ_2. We assume that λ_2 is simple. Then $Lcut(G)$ is defined as $Lcut(G) = Ncut(V^+(U_2) \cup V^0(U_2), V^-(U_2))$.

Example 12. Figure 8 shows some examples, where $Mcut(G) = Lcut(G)$ and $Mcut(G) \neq Lcut(G)$.

Proposition 24 ([16])**.** *Let $G = (V, E, w)$ be a weighted graph, W the weighted adjacency matrix of G, L the weighted difference Laplacian $L(G)$ of G, and A a subset of V. If vector*



(a) $Lcut(G) = Mcut(G)$ (b) $Lcut(R_{4,7}) = Mcut(R_{4,7})$

(c) $Mcut(R_{6,4})$ (d) $Lcut(R_{6,4})$

Figure 8: $Mcut(G)$ and $Lcut(G)$.

$y = (y_1, \ldots, y_n)^T \in \Re^n$ is defined as

$$y = \begin{cases} \sqrt{\frac{vol(V \setminus A)}{vol(A)}} & \text{if } v_i \in A, \\ -\sqrt{\frac{vol(A)}{vol(V \setminus A)}} & \text{if } v_i \in V \setminus A, \end{cases}$$

then

1. *$y^T L y = vol(V) \cdot Ncut(A, V \setminus A)$,*
2. *$y^T D y = vol(V)$ and*
3. *$(Dy)^T \vec{1} = 0$.*

Proof. 1.

$$\begin{aligned} y^T L y &= y^T D y - y^T W y \\ &= \sum_{i=1}^{n} d_i y_i^2 - \sum_{i,j} y_i w_{ij} y_j \\ &= \frac{1}{2}\left(\sum_{i=1}^{n} d_i y_i^2 - 2\sum_{i,j} y_i y_j w_{ij} + \sum_{j=1}^{n} d_j y_j^2\right) \\ &= \frac{1}{2}\sum_{i,j=1}^{n} w_{ij}(y_i - y_j)^2 \end{aligned}$$

This can be further reduced to,

$$\begin{aligned} &= \frac{1}{2}\sum_{i\in A, j\in(V\setminus A)} w_{ij}\left(\sqrt{\frac{vol(V\setminus A)}{vol(A)}} + \sqrt{\frac{vol(A)}{vol(V\setminus A)}}\right)^2 + \\ &\quad \frac{1}{2}\sum_{i\in(V\setminus A), j\in A} w_{ij}\left(-\sqrt{\frac{vol(A)}{vol(V\setminus A)}} - \sqrt{\frac{vol(V\setminus A)}{vol(A)}}\right)^2 \\ &= cut(A, V\setminus A)\left(\frac{vol(A)}{vol(V\setminus A)} + \frac{vol(V\setminus A)}{vol(A)} + 2\right) \\ &= cut(A, V\setminus A)\left(\frac{vol(A) + vol(V\setminus A)}{vol(V\setminus A)} + \frac{vol(A) + vol(V\setminus A)}{vol(A)}\right) \\ &= vol(V).Ncut(A, V\setminus A). \end{aligned}$$

2.

$$\begin{aligned} y^T D y &= \sum_{i=1}^{n} d_i y_i^2 = \sum_{i\in A} d_i y_i^2 + \sum_{i\in V\setminus A} d_i y_i^2 \\ &= \sum_{i\in A} d_i\left(\frac{vol(V\setminus A)}{vol(A)}\right) + \sum_{i\in(V\setminus A)} d_i\left(\frac{vol(A)}{vol(V\setminus A)}\right) \\ &= vol(A)\frac{vol(V\setminus A)}{vol(A)} + vol(V\setminus A)\frac{vol(A)}{vol(V\setminus A)} \\ &= vol(V). \end{aligned}$$

□

3.

$$\begin{aligned} (Dy)^T \vec{1} &= \sum_{i=1}^{n} d_i y_i \\ &= \sum_{i\in A} d_i \sqrt{\frac{vol(V\setminus A)}{vol(A)}} - \sum_{i\in V\setminus A} d_i \sqrt{\frac{vol(A)}{vol(V\setminus A)}} \\ &= vol(A)\sqrt{\frac{vol(V\setminus A)}{vol(A)}} - vol(V\setminus A)\sqrt{\frac{vol(A)}{vol(V\setminus A)}} \\ &= 0. \end{aligned}$$

□

By Proposition 24, we have

$$\begin{aligned} Ncut(A, V\setminus A) &= \frac{y^T L y}{y^T D y} \\ &= \frac{y^T (D - W) y}{y^T D y} \\ &= \frac{(D^{1/2}y)^T (I - D^{-1/2} W D^{-1/2})(D^{1/2}y)}{(D^{1/2}y)^T (D^{1/2}y)} \\ &= \frac{z^T \mathcal{L}(G) z}{z^T z}, \end{aligned}$$

where $z = D^{1/2}y$ and $\mathcal{L}(G) = I - D^{-1/2}WD^{-1/2}$. The least eigenvalue of $\mathcal{L}(G)$ is 0 and an eigenvector is $D^{1/2}\vec{1}$. Let λ_2 be the second eigenvalue of $\mathcal{L}(G)$. It is well known

$$\lambda_2 = \min\{\frac{z^T \mathcal{L}(G) z}{z^T z} \mid z \in \Re^n,\ z \perp D^{1/2}\vec{1}\}.$$

If z is a second eigenvector, then $\lambda_2 = \frac{z^T \mathcal{L}(G) z}{z^T z}$ and $z \perp D^{1/2}\vec{1}$. These results guide to consider relations between a set A attaining $Mcut(G) = Ncut(A, V\setminus A)$ and a set $V^+(U)$, where U is a second eigenvector of $\mathcal{L}(G)$. The set $V^+(U)$ is a good approximation of A.

5.2. The graph $R_{n,k}$

In this section, we review the formulae of $Mcut(R_{n,k})$ and conditions in Theorem 2, consider some properties of subsets A of $V(R_{n,k})$, which attains $Lcut(R_{n,k}) = Ncut(A, V\setminus A)$, and assign a condition of n and k to cause $Mcut(R_{n,k}) \neq Lcut(R_{n,k})$.

Let $R_{n,k} = (V, E)$, $V = \{v_i \mid 1 \le i \le 2(n+k)\}$, where

$$v_i = \begin{cases} x_i & (1 \le i \le n+k), \\ y_{i-(n+k)} & (n+k+1 \le i \le 2(n+k)). \end{cases}$$

We review subsets A_1, A_2 and $A_4(\alpha)$ defined in the proof of Theorem 2. That is

$$\begin{aligned} A_1 &= \{v_i \mid 1 \leq i \leq n+k\}, \\ A_2 &= \{v_i \mid 1 \leq i \leq n\}, \text{ and} \\ A_4(\alpha) &= \{v_i, v_{i+n+k} \mid 1 \leq i \leq n+\alpha\} \\ & \quad (1 \leq \alpha < k). \end{aligned}$$

For a vector $U = (u_1, u_2, \ldots, u_{2(n+k)}) \in \Re^{2(n+k)}$, we write $\bar{U} = (u_1, u_2, \ldots, u_{n+k}) \in \Re^{n+k}$. For a vector $\bar{U} = (u_1, u_2, \ldots, u_{(n+k)}) \in \Re^{n+k}$, we write $(\bar{U}, \bar{U}) \in \Re^{2(n+k)}$ as a vector $U = (u_1, u_2, \ldots, u_{2(n+k)}) \in \Re^{2(n+k)}$ such that $u_{i+(n+k)} = u_i$ $(1 \leq i \leq n+k)$. In this section, we consider an automorphism ϕ, where $\phi(v_i) = v_{i+n+k}$ to consider even and odd vectors.

Proposition 25. *If $U = (u_1, u_2, \ldots, u_{2(n+k)})$ is an eigenvector of $\mathcal{L}(R_{n,k})$ with an eigenvalue λ, then $\bar{U}$ is an eigenvector of $\mathcal{L}(P_{n,k})$ with an eigenvalue λ. Conversely, if $\bar{U} = (u_1, u_2, \ldots, u_{n+k})$ is an eigenvector of $\mathcal{L}(P_{n,k})$ with an eigenvalue λ, then $U = (\bar{U}, \bar{U})$ is an eigenvector of $\mathcal{L}(R_{n,k})$.*

Proof. If U is an even vector then we can write $U = (\bar{U}, \bar{U})$. The matrix $\mathcal{L}(R_{n,k})$ can be written as

$$\mathcal{L}(R_{n,k}) = \begin{pmatrix} \mathcal{L}_1 & C \\ C^T & \mathcal{L}_1 \end{pmatrix},$$

where $\mathcal{L}_1$ is the $(n+k) \times (n+k)$ principal sub matrix of $\mathcal{L}(R_{n,k})$ and $C = (c_{ij})$ is the $(n+k) \times (n+k)$ matrix such that

$$c_{ij} = \begin{cases} -\frac{1}{d_i} & \text{if } n+1 \leq i \leq n+k \text{ and } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

We notice that $\mathcal{L}_1 + C = \mathcal{L}(P_{n,k})$. If λ is an eigenvalue of $\mathcal{L}(R_{n,k})$ then $\mathcal{L}(R_{n,k})U = \lambda U$ can be written as,

$$\begin{pmatrix} \mathcal{L}_1 & C \\ C^T & \mathcal{L}_1 \end{pmatrix} \begin{pmatrix} \bar{U} \\ \bar{U} \end{pmatrix} = \lambda \begin{pmatrix} \bar{U} \\ \bar{U} \end{pmatrix}$$

This gives

$$\mathcal{L}_1 \bar{U} + C\bar{U} = \lambda \bar{U}.$$

This can be written as, $(\mathcal{L}_1 + C)\bar{U} = \mathcal{L}(P_{n,k})\bar{U} = \lambda \bar{U}$. Therefore λ is an eigenvalue of $\mathcal{L}(P_{n,k})$ and $\bar{U}$ is an eigenvector. Thus if U is an even vector of $\mathcal{L}(R_{n,k})$ with eigenvalue λ, then $\bar{U}$ is an eigenvector of $\mathcal{L}(P_{n,k})$ with the same eigenvalue. The converse also holds. □

Proposition 26. *Let $U = (u_1, u_2, \ldots, u_{(n+k)})$ be an eigenvector of $\mathcal{L}(P_{n,k})$ with a second smallest eigenvalue λ_2. Then there exists some $\alpha \in \mathbf{Z}^+$ such that $u_i \geq 0$ $(1 \leq i \leq \alpha)$ and $u_i < 0$ $(\alpha+1 \leq i \leq n+k)$ or $u_i < 0$ $(1 \leq i \leq \alpha)$ and $u_i \geq 0$ $(\alpha+1 \leq i \leq n+k)$.*

Proof. If $U = (u_1, u_2, \ldots, u_{n+k})$ is the second eigenvector of $\mathcal{L}(P_{n,k})$, then $U \perp D^{1/2}\vec{1}$. Then by Lemma 4, $V^+(U) \neq \emptyset$ and $V^-(U) \neq \emptyset$. Since λ_2 is simple, induced subgraphs by $V^+(U)$, $V^-(U)$, $V^+(U) \cup V^0(U)$ and $V^-(U) \cup V^0(U)$ are connected by the nodal domain theorem [5]. Thus there exists some $\alpha \in \mathbf{Z}^+$ as given in the proposition. □

Corollary 2. *If $\bar{U} = (u_1, u_2, \ldots, u_{n+k})$ is a first eigenvector of $\mathcal{L}(P_{n,k})$, then $U = (\bar{U}, \bar{U})$ is a first eigenvector of $\mathcal{L}(R_{n,k})$.*

□

Proposition 27. *Let λ_2 be the second smallest eigenvalue of $\mathcal{L}(R_{n,k})$, λ_2' the second smallest eigenvalue of $\mathcal{L}(P_{n,k})$, and $U = (u_1, u_2, \ldots, u_{2(n+k)})$ an eigenvector of $\mathcal{L}(R_{n,k})$ with λ_2. If U is an even vector then $\lambda_2 = \lambda_2'$. That is $\bar{U} = (u_1, u_2, \ldots, u_{n+k})$ is an second eigenvector of $\mathcal{L}(P_{n,k})$ with λ_2'.*

Proof. Since U is an even vector, $\bar{U}$ is an eigenvector of $\mathcal{L}(P_{n,k})$ with λ_2. So we have $\lambda_2' \leq \lambda_2$. We note $U \perp D^{1/2}(R_{n,k})\vec{1}$ and $\bar{U} \perp D^{1/2}(P_{n,k})\vec{1}$.

Let $U' = (u_1', u_2', \ldots, u_{n+k}')$ be a second eigenvector of $\mathcal{L}(P_{n,k})$ with λ_2'. Since (U', U') is an eigenvector of $\mathcal{L}(R_{n,k})$ with λ_2', we have $\lambda_2 \leq \lambda_2'$ and $\lambda_2 = \lambda_2'$. □

Let λ_2 be the second eigenvalue of $R_{n,k}$, U an eigenvector of $R_{n,k}$ with λ_2. Since λ_2 is simple, induced subgraphs by $V^-(U)$ and $V^+(U) \cup V^0(U)$ are connected by the nodal domain theorem [5]. Since U is an odd vector or an even vector, it is easy to show Lemma 20 and Lemma 21.

Lemma 20. *Let $U = (u_1, \ldots, u_{2(n+k)})$ be a second eigenvector of $\mathcal{L}(R_{n,k})$. If U is an odd vector then*

$$Lcut(R_{n,k}) = Ncut(A_1, V \setminus A_1).$$

□

Lemma 21. *Let $U = (u_1, \ldots, u_{2(n+k)})$ be a second eigenvector of $\mathcal{L}(R_{n,k})$. If U is an even vector then there exists α $(1 \leq \alpha < k)$ such that*

$$Lcut(R_{n,k}) = Ncut(A_4(\alpha), V \setminus A_4(\alpha)).$$

□

Proposition 28. *Let $G = R_{n,k}(n \geq 1, k \geq 2)$. If n and k belong to the following region R then $Mcut(G) < Lcut(G)$.*

$$\begin{aligned} R = \; & \{(n,k) \mid ((k \geq 4) \wedge (2 \mid k) \wedge (3 \mid n) \wedge \\ & (1 - \frac{1}{\sqrt{2}} - \frac{3k}{2} + \frac{3k}{\sqrt{2}} \leq n)) \vee \\ & (k = 2 \wedge (n \geq 2)) \vee (k = 3 \wedge (n \geq 3))\}. \end{aligned}$$

Proof. Let $G = (V, E)$, K_1, K_2, K_3 and K_4 are formulae defined in the Theorem 2. If $k \geq 2$ then $K_2 < K_3 < K_4 < K_1$. So if $(n,k) \in R$ then $Mcut(G) = Ncut(A_2, V \setminus A_2)$ and denoted by c_2 in the Theorem 2.

Let $U = (u_1, u_2, \ldots, u_{2(n+k)})$ be an eigenvector corresponding to the second smallest eigenvalue of $\mathcal{L}(R_{n,k})$. If U is an odd vector, then $Lcut(G) = Ncut(A_1, V \setminus A_1)$ by Lemma 20. So we have $Mcut(G) < Lcut(G)$ by Theorem 2.

If U is an even vector, then $Lcut(G) = Ncut(A_4(\alpha), V \setminus A_4(\alpha))$ for some α by Lemma 21. So we have $Mcut(G) < Lcut(G)$ by Theorem 2. □

Theorem 7. *Let $k \geq 3$, $\lambda_2(\mathcal{L}(P_{2k,k}))$, $\lambda_2(\mathcal{L}(P_{4k}))$, and $\lambda_2(\mathcal{L}(R_{2k,k}))$ the second eigenvectors of $\mathcal{L}(P_{2k,k})$, $\mathcal{L}(P_{4k})$, and $\mathcal{L}(R_{2k,k})$, respectively.*

1. *$\lambda_2(\mathcal{L}(P_{4k})) < \lambda_2(\mathcal{L}(P_{2k,k}))$.*
2. *$\lambda_2(\mathcal{L}(R_{2k,k})) < \lambda_2(\mathcal{L}(P_{4k}))$.*
3. *A second eigenvector U of $\mathcal{L}(R_{2k,k})$ is an odd vector.*
4. *The second eigenvalue of $\mathcal{L}(R_{2k,k})$ is simple.*
5. *$Mcut(R_{2k,k}) < Lcut(R_{2k,k})$.*

Proof. 1. Since $\lambda_2(\mathcal{L}(P_{4k})) = 1 - \cos\left(\frac{\pi}{4k-1}\right)$ by Proposition 13, we have $\lambda_2(\mathcal{L}(P_{4k})) < \lambda_2(\mathcal{L}(P_{2k,k}))$ by Proposition 22.

2. Let $A = (a_{ij})_{1\le i,j\le 4k}$ be the adjacency matrix of P_{4k}, $B = (b_{ij})_{1\le i,j\le 6k}$ be the adjacency matrix of $R_{2k,k}$, $d = (d_i)_{1\le i\le 4k}$, where $d_i = \sum_{j=1}^{4k} a_{ij}$, $e = (e_i)$ where $e_i = \sum_{j=1}^{6k} b_{ij}$, and $x = (x_i)_{1\le i\le 4k}$ an eigenvector of $\mathcal{L}(P_{4k})$ corresponding to $\lambda_2(\mathcal{L}(P_{4k}))$ with $x^T x = 1$. We note that $d^{\frac{1}{2}}\vec{1} \perp x$ and $\lambda_2(\mathcal{L}(P_{4k})) = x^T\mathcal{L}(P_{4k})x = \frac{1}{2}\sum_{i=1}^{4k}\sum_{j=1}^{4k}\left(\frac{x_i}{\sqrt{d_i}} - \frac{x_j}{\sqrt{d_j}}\right)^2 a_{ij}$.
Let

$$y_i = \begin{cases} x_i & (1 \le i \le 2k) \\ 0 & (2k+1 \le i \le 3k,\ 5k+1 \le i \le 6k) \\ x_{7k-i+1} & (3k+1 \le i \le 5k) \end{cases}$$

and consider a vector $y = (y_i)_{1\le i\le 6k}$. Since x is a second eigenvector of $\mathcal{L}(P_{4k})$, we have $\sum_{i=1}^{6k} y_i^2 = \sum_{i=1}^{4k} x_i^2 = 1$, $\sum_{i=1}^{6k}\sqrt{e_i}y_i = \sum_{i=1}^{4k}\sqrt{d_i}x_i = 0$, and $x_{2k} = -x_{2k+1} \neq 0$. So we have $y^T y = 1$, $e^{\frac{1}{2}}\vec{1} \perp y$, and

$$\begin{aligned}
\lambda_2(R_{2k,k}) &= \inf_{e^{\frac{1}{2}}\vec{1}\perp u} \frac{u^T\mathcal{L}(R_{2k,k})u}{u^T u} \\
&\le y^T\mathcal{L}(R_{2k,k})y \\
&= \frac{1}{2}\sum_{i=1}^{6k}\sum_{j=1}^{6k}\left(\frac{y_i}{\sqrt{e_i}} - \frac{y_j}{\sqrt{e_j}}\right)^2 b_{ij} \\
&= \frac{1}{2}\sum_{i=1}^{4k}\sum_{j=1}^{4k}\left(\frac{x_i}{\sqrt{d_i}} - \frac{x_j}{\sqrt{d_j}}\right)^2 a_{ij} - \left(\frac{x_{2k}}{\sqrt{d_{2k}}} - \frac{x_{2k+1}}{\sqrt{d_{2k+1}}}\right)^2 \\
&\quad + \left(\frac{y_{2k}}{\sqrt{e_{2k}}} - \frac{y_{2k+1}}{\sqrt{e_{2k+1}}}\right)^2 + \left(\frac{y_{5k}}{\sqrt{e_{5k}}} - \frac{y_{5k+1}}{\sqrt{e_{5k+1}}}\right)^2 \\
&= \lambda_2\left(\mathcal{L}(P_{4k})\right) - \left(\frac{2x_{2k}}{\sqrt{d_{2k}}}\right)^2 + \left(\frac{y_{2k}}{\sqrt{e_{2k}}}\right)^2 + \left(\frac{y_{5k}}{\sqrt{e_{5k}}}\right)^2 \\
&= \lambda_2\left(\mathcal{L}(P_{4k})\right) - 2\left(\frac{x_{2k}}{\sqrt{d_{2k}}}\right)^2 \\
&< \lambda_2\left(\mathcal{L}(P_{4k})\right)
\end{aligned}$$

3. If a second eigenvector U of $\mathcal{L}(R_{2k,k})$ corresponding to $\lambda_2(\mathcal{L}(R_{2k,k}))$ is an even vector, then $\lambda_2(\mathcal{L}(R_{2k,k})) = \lambda_2(\mathcal{L}(P_{2k,k}))$ by Proposition 23. But it contradicts that $\lambda_2(\mathcal{L}(R_{2k,k})) < \lambda_2(\mathcal{L}(P_{2k,k}))$ induced by 1. and 2. So we have a second eigenvector U of $\mathcal{L}(R_{2k,k})$, which is an odd vector.

4. By 3. and Proposition 23, 3. and 4., $\lambda_2(\mathcal{L}(R_{2k,k}))$ is simple.

5. Since the second eigenvector of $R_{2k,k}$ is an odd vector, $Lcut(R_{2k,k}) = Ncut(A_1, V \setminus A_1)$ by Lemma 20. Thus we have $Mcut(R_{2k,k}) < Lcut(R_{2k,k})$ by Theorem 2.

□

6. Conclusion

We presented a survey of the known results associated with difference, normalized, and signless Laplacian matrices. We also stated upper and lower bounds for the difference and normalized Laplacian matrices using isoperimetric numbers and the Cheeger constant. We gave a uniform proof for the eigenvalues and eigenvectors of paths and cycles on the basis of all three Laplacian matrices using circulant matrices, and presented an alternate proof for finding the eigenvalues of the adjacency matrix of cycles and paths using Chebyshev polynomials. We also introduced concrete formulae for $Mcut(G)$ for some classes of graphs. Then, we established characteristic polynomials for the normalized Laplacian matrices $\mathcal{L}(P_{n,k})$ and $\mathcal{L}(R_{n,k})$. Finally, we presented counter example graphs based on $R_{n,k}$, where $Mcut(G)$ and $Lcut(G)$ produce different clusters. In particular, we established criteria for $Mcut(G)$ and $Lcut(G)$ to have different values.

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